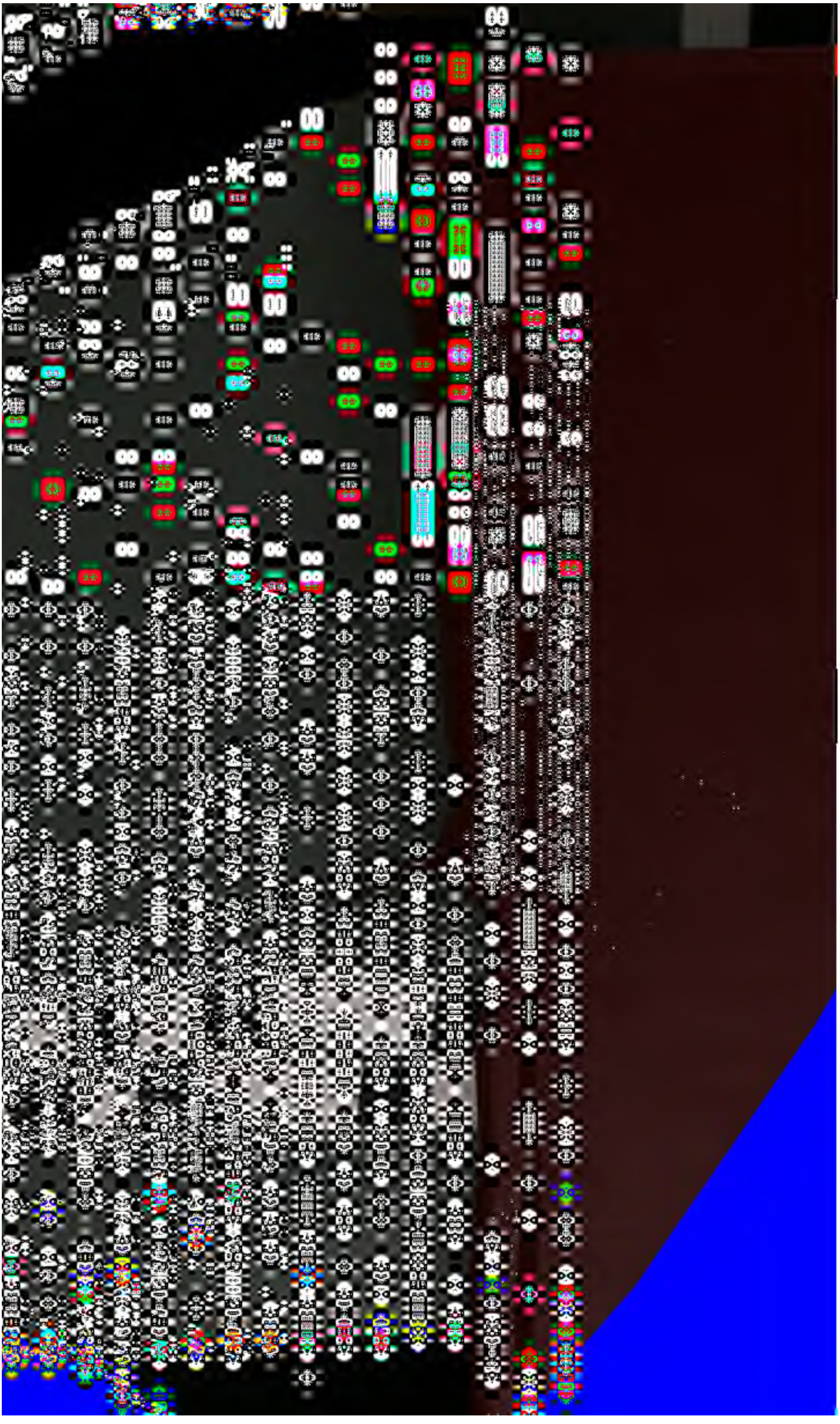

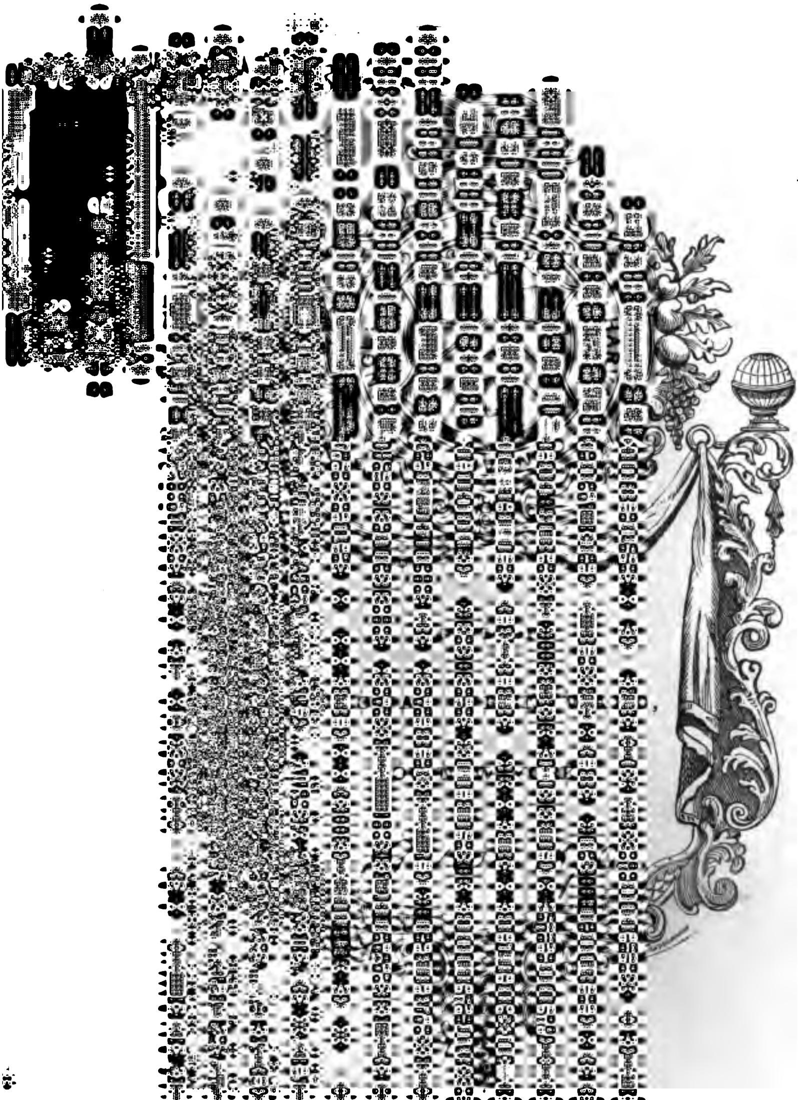

5/31

A COLLECTION

OF

EXAMPLES AND PROBLEMS

IN

PURE AND MIXED

MATHEMATICS,

WITH

ANSWERS AND OCCASIONAL HINTS.

BY

THE REV. ALFRED WRIGLEY, M.A., F.R.A.S. &c.,
OF ST. JOHN'S COLLEGE, CAMBRIDGE;
FIRST ASSISTANT PROFESSOR OF MATHEMATICS AND CLASSICS IN THE
LATE ROYAL INDIAN MILITARY COLLEGE, ADDISCOMBE.

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1862.

aid of tables, in which the logarithms consist of six figures. The Logarithmic Tables here used, were edited by the Rev. J. Cape, M.A., &c., Professor of Mathematics and Classics in the East India Company's Military College, Addiscombe.

As in many calculations, logarithms are used rather for convenience than from necessity, the answers obtained will slightly vary, (1) according as logarithms are used or not, (2) according as the tabulated logarithm consists of five, six, or seven figures, more or less, and (3) according to the extent of the use made of 'proportional parts.' This variation is generally very small, and an indication of the cause of it will be sufficient to explain the disparity between the answers to some questions obtained on different occasions or by different persons.

Great assistance has been derived from Barlow's Tables of Squares, Cubes, Square Roots, &c., edited by Professor De Morgan, 1840.

November 15, 1857.

ADVERTISEMENT.

The Sixth Edition is a reprint of the Fifth corrected.

January 1862.

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THEORY OF EQUATIONS.

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EXAMPLES AND PROBLEMS

IN

PURE AND MIXED MATHEMATICS.

ARITHMETIC.

VULGAR FRACTIONS.

Ex. 1. Reduce to their lowest terms—

1. $\frac{60}{144}$; $\frac{96}{324}$; $\frac{420}{5400}$; $\frac{598}{1058}$; $\frac{1407}{2211}$.

2. $\frac{1792}{2048}$; $\frac{6409}{7395}$; $\frac{2433}{13787}$; $\frac{8398}{29393}$; $\frac{19527}{23667}$.

3. $\frac{23760}{26136}$; $\frac{527751}{645029}$; $\frac{100110}{31866}$; $\frac{1099901}{1067803}$.

Ex. 2. Reduce to simple fractions—

1. $1\frac{11}{12}$; $55\frac{5}{9}$; $1805\frac{3}{8}$; $20\frac{119}{120}$.

2. $3\frac{13}{15}$; $5\frac{11}{17}$; $8\frac{15}{19}$; $9\frac{17}{18}$; $13\frac{20}{21}$.

Ex. 3. Express as simple fractions—

1. $\frac{1}{2}$ of $\frac{7}{8}$ of 3; $\frac{2}{3}$ of $\frac{1}{4}$ of 9; $\frac{3}{5}$ of $\frac{5}{6}$ of 8.

2. $\frac{2}{5}$ of $\frac{3}{11}$ of $6\frac{2}{3}$; $\frac{3}{7}$ of $2\frac{1}{9}$ of $2\frac{4}{19}$; $1\frac{1}{2}$ of $2\frac{1}{2}$ of $3\frac{1}{5}$ of $\frac{1}{20}$.

3. $\frac{12}{13}$ of $2\frac{2}{3}$ of $1\frac{1}{25}$ of $1\frac{11}{64}$; $\frac{7}{8}$ of $\frac{3}{4}$ of $\frac{2}{5}$ of $\frac{5}{7}$ of $\frac{16}{9}$.

Ex. 4. Find the sum of—

1. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$; $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\frac{7}{8}$.

Ex. 4. Find the sum of—

2. $\frac{2}{3}+\frac{1}{6}+\frac{5}{9}+\frac{11}{12}$; $\frac{1}{8}+\frac{1}{63}+\frac{1}{560}+\frac{1}{5040}$.
3. $3\frac{1}{3}+\frac{5}{6}+4\frac{1}{9}+2$; $\frac{2}{7}+5\frac{2}{11}+1\frac{1}{3}$ of $2\frac{1}{2}+6\frac{1}{4}$.
4. $\frac{2}{5}+\frac{35}{80}+\frac{14}{100}+\frac{3}{140}+\frac{3}{2800}$ $8\frac{1}{2}+9\frac{2}{3}+10\frac{3}{4}+11\frac{4}{5}+12\frac{5}{6}$.

Ex. 5. Find the value of—

1. $\frac{1}{3}+\frac{1}{5}+\frac{6}{7}-\frac{4}{15}$; $1-\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$.
2. $\frac{4}{17}+\frac{7}{50}+\frac{21}{850}-\frac{11}{90}$; $\frac{15}{16}-\frac{14}{15}+\frac{13}{14}-\frac{11}{12}$.
3. $4-\frac{2}{5}-\frac{4}{5}-\frac{3}{4}+\frac{5}{4}-\frac{1}{2}$; $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{7}{6}-\frac{5}{8}+\frac{5}{6}$.
4. $(1\frac{2}{5}$ of $3\frac{5}{9})-2\frac{1}{15}$; $(4\frac{1}{3}$ of $5\frac{3}{8})-(2\frac{1}{7}$ of $\frac{14}{5})+7\frac{2}{9}$.

Ex. 6. Find the product of—

1. $\frac{5}{8}\times\frac{14}{15}\times 2\frac{1}{3}$; $2\frac{1}{5}\times 3\frac{1}{3}\times\frac{30}{38}$; $\frac{7}{3}\times 11\frac{1}{9}\times 4\frac{3}{100}\times 27$.
2. $\frac{7}{18}\times\frac{11}{10}\times\frac{27}{14}\times\frac{5}{2}\times\frac{16}{7}$; $3\frac{2}{3}\times\frac{1}{7}\times(\frac{3}{5}$ of $\frac{3}{4})$; $\frac{93}{4600}\times\frac{125}{662}$.
3. $(\frac{1}{3}$ of $\frac{9}{8}$ of $2\frac{2}{3})\times 15\frac{2}{7}$; $(1\frac{1}{4}$ of $\frac{5}{7}$ of $6\frac{1}{10})\times(\frac{14}{13}$ of $2\frac{4}{11})\times\frac{33}{49}$.

Ex. 7. Find the quotient of—

1. $4\div\frac{2}{5}$; $\frac{3}{7}\div\frac{13}{14}$; $\frac{22}{35}\div\frac{33}{25}$; $3\frac{1}{2}\div\frac{1}{2}$; $\frac{24}{35}\div 3\frac{3}{7}$.
2. $147\frac{7}{12}\div 12\frac{3}{5}$; $15\frac{3}{5}\div 7\frac{4}{5}$; $(\frac{2}{3}$ of $\frac{5}{8})\div\frac{7}{18}$; $42\div(1\frac{2}{3}$ of $\frac{5}{7})$.
3. $(\frac{7}{8}$ of $6)\div(\frac{3}{4}$ of $\frac{6}{7}$ of $\frac{11}{12})$; $(\frac{7}{6}$ of $\frac{1}{2}$ of $\frac{3}{10})\div(\frac{6}{11}$ of $\frac{7}{8}$ of $\frac{9}{2})$.
4. $(3\frac{3}{7}\times 2\frac{11}{12})\div(8\frac{9}{14}$ of $\frac{1}{33})$; $(\frac{1}{7}\times\frac{4}{9}\times 3)\div(\frac{6}{11}\times\frac{8}{9}\times 4)$.

Ex. 8. Find the value of—

1. $1+2\times\frac{4}{3}+3\times\frac{16}{9}$; $7\frac{4}{5}-7\times\frac{4}{5}$; $\frac{2}{5}(6\frac{2}{3}+2\frac{1}{2})$.
2. $(2\frac{1}{2}+\frac{1}{6})\div(3\frac{2}{3}-\frac{1}{8})$; $(\frac{2247}{1017}\div\frac{903}{1017})\times(\frac{774}{615}\div\frac{1926}{565})$.
3. $\dfrac{\frac{3}{7}\times 1\frac{2}{5}\times 12\frac{1}{2}}{6\frac{2}{3}}$; $\dfrac{4\frac{1}{7}-2\frac{1}{4}}{6\frac{1}{2}-2\frac{1}{7}}$; $\dfrac{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}}{1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}}$.
4. $\dfrac{\frac{13}{40}-\frac{14}{41}}{\frac{15}{42}-\frac{16}{43}}$; $\dfrac{\frac{13}{21}\times\frac{1}{2}-\frac{11}{14}\times\frac{1}{3}}{\frac{16}{21}\times\frac{1}{2}-\frac{13}{14}\times\frac{1}{3}}$; $\dfrac{2\frac{1}{4}-\frac{2}{3}\times 1\frac{5}{6}}{\frac{1}{5}\times 3\frac{1}{3}+\frac{13}{36}}$.
5. $\dfrac{2\frac{1}{2}\times 2\frac{1}{2}\times 2\frac{1}{2}-1}{2\frac{1}{2}\times 2\frac{1}{2}-1}$; $\dfrac{4\frac{1}{3}\times 4\frac{1}{3}\times 4\frac{1}{3}-4}{4\frac{1}{3}\times 4\frac{1}{3}-4}$; $\dfrac{6\frac{1}{4}\times 6\frac{1}{4}\times 6\frac{1}{4}-8}{6\frac{1}{4}\times 6\frac{1}{4}-4}$.
6. $\dfrac{\frac{2}{19}+\frac{1}{3}}{3-\frac{1}{3}}\times(\frac{1}{3}+\frac{1}{5})$; $\frac{18}{17}(1-\frac{64}{81})+\frac{8}{11}\times\frac{1}{6}\times(\frac{1}{2}\times\frac{5}{12})$.
7. $\frac{2}{7}\times\dfrac{1-\frac{2}{7}}{2}+\frac{4}{5}\times\frac{1}{10}+\frac{3}{5}\times(\frac{1}{2}+\frac{11}{14})+\frac{3}{70}\times(\frac{2}{7}+\frac{4}{5})$.
8. $\left(\dfrac{3\frac{1}{3}}{7}+\dfrac{2}{10\frac{1}{2}}-\frac{5}{18}\text{ of }\frac{4}{7}\right)\times 1\frac{1}{8}$; $\dfrac{2\frac{1}{2}}{3\frac{1}{4}}+\dfrac{1\frac{1}{2}-\frac{5}{6}}{1\frac{1}{4}+\frac{5}{6}}-1\frac{2}{39}$.
9. $\left(2\frac{3}{4}+\dfrac{5}{3\frac{4}{5}}\text{ of }3\frac{1}{2}-\dfrac{2\frac{1}{3}}{3\frac{1}{2}}\right)\div 1\frac{77}{228}$.

Ex. 8. Find the value of—

10. $\dfrac{2}{3+\dfrac{4}{5+\dfrac{6}{7}}}$; $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4+\dfrac{1}{5}}}}$; $7\frac{3}{13}$ of $\dfrac{1}{10+\dfrac{1}{3+\dfrac{1}{30}}}$.

Ex. 9. Find the value of—

1. $\frac{3}{14}$ of 10*s*. 6*d*.; $\frac{4}{5}$ of £3 12*s*.; $\frac{367}{1296}$ of 27*s*.
2. $\frac{3}{8}$ of a cwt.; $\frac{1}{2}$ of $1\frac{1}{5}$ of a lb. troy; $\frac{41}{726}$ of a lb. troy.
3. $\frac{63}{160}$ of a ton; $\frac{5}{8}$ of an acre; $\frac{5}{7}$ of $\frac{3}{4}$ of a day.
4. £2 15*s*. 8*d*. $\times \frac{5}{6}$; £23 10*s*. 11*d*. $\times 2\frac{1}{11}$; £1 13*s*. 9*d*. $\times 1\frac{2}{3}$.
5. (£75 13*s*. $9\frac{1}{5}$*d*.) $\times \frac{15}{4}$; (£525 14*s*. $6\frac{5}{8}$*d*.) $\times \frac{42}{11}$.
6. (£18 17*s*. 0*d*. $\times 2\frac{7}{16}$); (£40 11*s*. $6\frac{3}{8}$*d*.) $\times 57\frac{1}{20}$.
7. $\frac{3}{7}$ of £5 18*s*. 5*d*.; $\frac{2}{5}$ of $\frac{3}{7}$ of £16 8*s*. $1\frac{1}{2}$*d*.
8. 4 miles 3 fur. 37 poles $4\frac{1}{5}$ yd. $\times 5\frac{5}{7}$; 2 cwt. 3 qr. $\times \frac{4}{11}$.
9. £$4\frac{1}{6}$ + $11\frac{1}{3}$*s*. + $7\frac{9}{12}$*d*.; £$\frac{2}{9}$ + $\frac{4}{11}$*s*. + ($\frac{4}{9}$ of 21*s*.)
10. $\frac{3}{4}$ of 21*s*. + $\frac{3}{8}$ of 5*s*. + $\frac{3}{5}$ of 7*s*. 6*d*. − $\frac{3}{4}$ of 2*d*.
11. $\frac{2}{3}$ of 10*s*. 6*d*. + $\frac{7}{8}$ of 27*s*. − $\frac{6}{11}$ of 6*s*. 8*d*.
12. $\frac{3}{4}$ of $10\frac{1}{2}$*d*. + $1\frac{2}{3}$ of $\frac{1}{9}$ of 5*s*. + $\frac{3}{8}$ of $1\frac{1}{4}$ of £1.
13. $\dfrac{15\frac{3}{5}}{7\frac{4}{5}}$ of £1 + $\frac{1}{5}$ of £140 10*s*. 6*d*. + $\frac{3}{5}$ of 21*s*.
14. $1\frac{3}{4}$ of $3\frac{1}{2}$ of £1 7*s*. + $\frac{5}{8}$ of 13*s*. 4*d*. − $\frac{2}{7}$ of $1\frac{5}{9}$ of $6\frac{2}{3}$*s*.
15. $7\frac{2}{5}$ of $365\frac{1}{4}$ days + $3\frac{9}{10}$ of $\frac{5}{6}$ wk. + $\frac{3}{4}$ of $5\frac{5}{9}$ hr.

Ex. 10. Find the value of—

1. £9 9*s*. $7\frac{5}{16}$*d*. $\div 3\frac{3}{4}$; £20 18*s*. $2\frac{157}{310}$*d*. $\div 12\frac{10}{12}$.
2. £160 4*s*. $8\frac{1}{4}$*d*. $\div \frac{1}{5}$ of £1 10*s*. $6\frac{1}{4}$*d*.; 3 wk. 4 d. $\div 1\frac{2}{3}$.
3. 2A. 3R. 5P. $\div 4\frac{7}{15}$; 1416A. 2R. 16P. $\div \frac{1}{8}$ of 4A. 5R. 27P.
4. $\frac{5}{14}$ of $1\frac{1}{3}$ of £2 16*s*. 3*d*. $\div \frac{4}{49}$; $2\frac{2}{5}$ of £8 14*s*. $2\frac{7}{12}$*d*. $\div 8\frac{4}{5}$.
5. 5 cwt. 2 qr. 14 lb. $\div 1\frac{2}{3}$; 3 m. 7 fur. 110 yd. $\div \frac{11}{12}$.

Ex. 11. Reduce—

1. 6*s*. 8*d*. to the fr. of £1; 8*s*. 2*d*. to the fr. of 21*s*.
2. 15*s*. $9\frac{1}{2}$*d*. to the fr. of £1; £2 17*s*. $7\frac{1}{5}$*d*. to the fr. of £3 12*s*.
3. £7 13*s*. $1\frac{1}{2}$*d*. to the fr. of £2 6*s*. $10\frac{1}{2}$*d*.; $2\frac{1}{2}$ guin. to fr. of £$1\frac{3}{4}$.
4. 5 yd. 1 ft. to the fr. of a mile.
5. 2 fur. 97 yd. $2\frac{1}{3}$ ft. to the fr. of a mile.

Ex. 11. Reduce—

6. 3 qr. 3 lb. 1 oz. $12\frac{4}{5}$ dr. to the fr. of a cwt.
7. 19 lb. to the fr. of 2 qr. $5\frac{1}{2}$ lb.
8. 13 wk. 5 d. $6\frac{1}{2}$ hr. to the fr. of $365\frac{1}{4}$ days.
9. 3 R. 15 P. to the fr. of an acre.
10. $3\frac{1}{2}$ crowns to the fr. of $7\frac{1}{8}$ guin.; $5\frac{1}{5}$ sq. yd. to the fr. of $1\frac{5}{8}$ P.

Ex. 12. Reduce—

1. $\frac{5}{7}d.$ to the fr. of £1; $2\frac{1}{4}s.$ to the fr. of a guinea.
2. $\frac{2}{5}$ of $2s.\ 4\frac{1}{2}d.$, and $\frac{3}{7}$ of $1s.\ 5\frac{1}{2}d.$, each to the fr. of $2s.\ 6d.$
3. $\frac{5}{7}$ of $1s.\ 2d.$ to the fr. of $5s.\ 10d.$; $\frac{3}{5}$ of $6s.\ 8d.$ to the fr. of $5s.$
4. $\frac{1}{1344}$ of a guin. to the fr. of $\frac{1}{4}d.$
5. $\frac{5}{8}$ of $1s.\ 9d.$ to the fr. of $3s.\ 4d.$
6. $3\frac{1}{7}$ of £4 $11s.$ to the fr. of $13\frac{1}{3}$ guineas.
7. $\frac{2}{3}$ of $1\frac{1}{5}$ of $9s.\ 7d.$ to the fr. of £$2\frac{7}{8}$.
8. $2\frac{5}{7}$ of 14 lb. to the fr. of 3 qr. 10 lb.
9. $1\frac{1}{4}$ sq. yd. to the fr. of 2 ft. 5 in.
10. $7\frac{3}{5}$ of 3 A. 2 R. 5 P. to the fr. of $11\frac{1}{3}$ acres.
11. $31\frac{1}{12}$ of 3d. 7 hr. to the fr. of 3 weeks.
12. $\frac{7}{8}$ of £1 $-\frac{7}{9}$ of a guin. to the fr. of a crown.
13. $\dfrac{3\frac{1}{9}}{1\frac{2}{13}} \times \{(\frac{19}{120}$ of £1$) - (\frac{7}{48}$ of $1s.)\}$ to the fr. of $27s.$

Ex. 13.

1. What part of $4\frac{1}{2}$ guineas is $5\frac{5}{8}$ of $\frac{2}{15}$ of £14?
2. What part of $1\frac{1}{3}$ roods is $25\frac{9}{11}$ poles; of 3 wk. 4 d. is $2\frac{4}{5}$ min.?
3. What sum is the same fr. of $5s.$ that $2s.\ 9\frac{3}{5}d.$ is of $21s.$?

DECIMAL FRACTIONS.

Ex. 14. Find the product—

1. Of $120{\cdot}5 \times 41{\cdot}76$; $375{\cdot}4 \times {\cdot}057$.
2. Of ${\cdot}47 \times {\cdot}0008$; ${\cdot}000476 \times {\cdot}0078$; $573{\cdot}005 \times {\cdot}000754$.
3. Of $814{\cdot}632 \times {\cdot}0378$; $91{\cdot}78 \times {\cdot}381$.
4. Of $3{\cdot}04 \times {\cdot}201 \times {\cdot}0152$; $101{\cdot}5 \times 1{\cdot}015 \times {\cdot}01015$.

Ex. 15. Find the quotient—

1. Of $1735{\cdot}5 \div 6{\cdot}5$; $6{\cdot}7288 \div 647$; $6 \div {\cdot}008$.
2. Of $3{\cdot}1 \div {\cdot}0025$; $2{\cdot}86 \div {\cdot}013$; $4{\cdot}8 \div {\cdot}00016$.

Ex. 15. Find the quotient—

3. Of $.07504 \div 23.45$; $.00044408 \div .0112$.
4. Of $365 \div .18349$; $1708.4592 \div .0072$.
5. Of $738.0964 \div .023$; $.10724 \div .003125$.

Ex. 16. Reduce to decimals—

1. $\frac{2}{25}$; $\frac{19}{125}$; $\frac{14}{5}$; $\frac{1}{800}$; $\frac{1}{128}$.
2. $\frac{5}{8}$; $\frac{13}{1000}$; $\frac{321}{625}$; $\frac{56}{1250}$; $13\frac{17}{50}$.
3. $7 \div 3\frac{1}{5}$; $(7\frac{1}{2} \text{ of } \frac{1}{25}) + \frac{17}{25}$; $3\frac{1}{19} \text{ of } 5\frac{1}{13} \text{ of } \frac{10}{29}$.
4. $\frac{47\frac{5}{8}}{94} \text{ of } \frac{11\frac{3}{4}}{7\frac{1}{2}}$; $\frac{7\frac{3}{4}}{9} \text{ of } \frac{2\frac{1}{2}}{2\frac{7}{9}} \text{ of } \frac{20}{31}$.
5. $\frac{15\frac{1}{2}}{32} \text{ of } \frac{11}{31\frac{1}{4}}$; $3 \text{ of } \frac{2}{19} \text{ of } 1\frac{1}{73}$.
6. $\frac{1}{3}$; $\frac{1}{7}$; $\frac{1}{15}$; $\frac{169}{900}$.
7. $\frac{13}{99}$; $\frac{129}{55}$; $\frac{111}{22}$; $\frac{4111}{33300}$.
8. $\frac{189}{925}$; $\frac{11}{21}$; $\frac{5}{31}$.

Ex. 17. Find the value of—

1. $581.4\dot{3} + .5\dot{3}\dot{1} + 8.\dot{0}\dot{1} + 1\dot{9}.04 + 307.5$.
2. $2.\dot{7} - .9\dot{1}\dot{3}$; $27.04\dot{2}769\dot{4} - 9.\dot{9}4\dot{6}$; $4 - 3.\dot{1}4285\dot{7}$.
3. $28.4\dot{3} \times 1.2\dot{4}$; $3.\dot{8}\dot{1} \times .\dot{1}0\dot{3}$; $5.\dot{0}9\dot{7} \times 12$.
4. $6.7\dot{6} \times .0\dot{5}$; $303.75 \times .1\dot{7}$; $4.58\dot{3} \times .58\dot{1}$.
5. $12 \div 1728$; $16 \div 960$; $.\dot{3} \div .\dot{0}\dot{9}$.
6. $.\dot{0}\dot{9} \div .\dot{0}\dot{4}$; $.042 \div .03\dot{6}$; $234.\dot{6} \div .\dot{7}$.
7. $.\dot{0}\dot{4} \div .\dot{7}6923\dot{0}$; $7 \div .\dot{1}4285\dot{7}$; $(2.8 \text{ of } 2.\dot{2}\dot{7}) \div 1.1\dot{3}\dot{6}$.
8. $25.21\dot{3} \div .40\dot{6}$; $(.\dot{3}\dot{6} \div 1.78) \times 5.9\dot{3} \div .072$.
9. $\left(\frac{2.375}{6.\dot{3}} \text{ of } \frac{8.\dot{8}}{.0625}\right) \div \left(\frac{17.\dot{7}}{11.35} \text{ of } \frac{4}{7}\right)$.
10. $\left(\frac{3.\dot{5} - 1.8\dot{3}}{4.\dot{1} + 5.\dot{8}} \text{ of } \frac{7.25 \text{ of } 1.2}{3.25}\right) + \frac{3.1 \text{ of } .1\dot{0}\dot{1}}{2.\dot{1}\dot{5}}$.

Ex. 18. Find the fractional values—

1. Of $.25$; $.725$; $.1634$.
2. Of 1.36; 3.582; 15.26.
3. Of $.085$; $.00875$; $.00023535$.
4. Of $.\dot{2}$; $.\dot{2}\dot{7}$; $.\dot{2}\dot{4}$; $1.\dot{7}\dot{2}$.

Ex. 18. Find the fractional values—

5. Of $\cdot 0\dot{5}\dot{1}$; $\quad \cdot 3\dot{1}4\dot{8}$; $\quad \cdot 011\dot{3}\dot{6}$.
6. Of $5\cdot 18\dot{7}$; $\quad \cdot 0087\dot{6}\dot{3}$; $\quad 11\cdot 2\dot{8}\dot{7}$.
7. Of $(12\cdot 5 + 1\cdot 25) \div (12\cdot 5 - 1\cdot 25)$.

Ex. 19. Find the value—

1. Of $13\frac{1}{4} + 1\frac{2}{5} + 8\frac{3}{125} - 4\frac{1}{8}$ by decimals, and by vulgar fractions.
2. Of $2\cdot 25 + 3\cdot 5\dot{7} + 1\cdot 375 + \cdot 3\dot{2}$; $\;101\cdot 5\dot{4} - 98\cdot 246\dot{4}$;

   $5\frac{1}{8} \times 2\cdot\dot{3} \times 3\cdot 6$; $\;\dfrac{51\cdot 25}{3\cdot 75}$; and the product of the results.
3. Of $1 + \dfrac{1}{1} + \dfrac{1}{1.2} + \dfrac{1}{1.2.3} + \dfrac{1}{1.2.3.4} + \dfrac{1}{1.2.3.4.5} + \dfrac{1}{1.2.3.4.5.6}$.
4. Of $2\left\{\dfrac{1}{5} + \dfrac{1}{3 \times 5^3} + \dfrac{1}{5 \times 5^5} + \dfrac{1}{7 \times 5^7} + \dfrac{1}{9 \times 5^9}\right\}$.
5. Of $2\left\{\dfrac{1}{3} + \dfrac{1}{3 \times 3^3} + \dfrac{1}{5 \times 3^5} + \dfrac{1}{7 \times 3^7} + \dfrac{1}{9 \times 3^9}\right\}$.
6. Of $4\left\{\dfrac{1}{5} - \dfrac{1}{3 \times 5^3} + \dfrac{1}{5 \times 5^5} - \dfrac{1}{7 \times 5^7} + \dfrac{1}{9 \times 5^9} - \dfrac{1}{11 \times 5^{11}}\right\} - \dfrac{1}{239}$.

Ex. 20. Reduce—

1. $4\frac{1}{2}$*d.* to the dec. of 1*s.*; $\quad$ 2*s.* $11\frac{3}{4}$*d.* to the dec. of £1.
2. 18*s.* $4\frac{1}{2}$*d.* to the dec. of £1; $\quad$ $1\frac{1}{2}$*d.* to the dec. of £1.
3. 4*s.* $7\frac{9}{11}$*d.* to the dec. of £1; $\quad$ 8*s.* $7\frac{1}{4}$*d.* to the dec. of 21*s.*
4. 14*s.* $6\frac{1}{2}$*d.* to the dec. of 27*s.*; $\quad$ $\frac{3}{5}$ of 21*s.* to the dec. of £1.
5. 5*s.* $4\frac{1}{2}$*d.* to the dec. of £1 7*s.* 6*d.*
6. £2 15*s.* 9*d.* to the dec. of 3*s.* $8\frac{1}{2}$*d.*
7. 14*s.* $9\frac{3}{4}$*d.* to the dec. of $2\frac{1}{2}$ guineas.
8. $5\frac{1}{4}$ guineas to the dec. of £50.
9. 11 dwt. to the dec. of a lb. troy.
10. 10 drams to the dec. of a lb. avoird.
11. 3 qr. 3 lb. 1 oz. $12\frac{4}{5}$ dr. to the dec. of a cwt.
12. 6 furlongs to the dec. of a league.
13. 5 lb. 10 oz. 3 dwt. 15 gr. troy, to the dec. of a cwt. avoird.
14. 1 cwt. 3 qr. 14 lb. to the dec. of $1\frac{1}{4}$ tons.
15. 12 min. to the dec. of an hour.
16. 3 R. 10 P. to the dec. of $2\frac{1}{2}$ acres.

Ex. 20. Reduce—

17. 5 sq. ft. 32 in. to the dec. of $1\frac{1}{4}$ sq. yd.
18. 3 wk. 4 d. 5 h. 6 m. 7 s. to the dec. of 4 weeks.
19. 5 days 9 h. 7 s. to the dec. of 2 wk. 15 h.

Ex. 21. Find the value of—

1. ·785 of £1; 3·465 of £1; ·75435 of 1*s.*
2. ·878125 of £1; ·97216 of £1; 1·7962 of £1.
3. ·375 of a guin.; ·72708 of £1 7*s.*; 6·25 of 6*s.* 8*d.*
4. 1·46875 of £3 4*s.* 6*d.*; 7·125 of $3\frac{1}{2}$ guin.; ·03125 of £25.
5. ·00243 of a lb. troy; ·0396 of a lb. avoird.; ·35 of 2 qr. 17 lb.
6. ·475 of a yd.; 27·138 of 2 m. 450 yd.; ·04535 of a mile.
7. ·05 of an acre; 1·3875 of 2 sq. ft. 95 in.; ·255 of a league.
8. 4·2$\dot{7}$ of 5*s.* $10\frac{1}{2}$*d.*; ·089$\dot{2}$8571$\dot{4}$ of 7*s.*; 2·3$\dot{5}$ of 3*s.*
9. ·013$\dot{8}$ of £4 14*s.* 6*d.*; 2·$\dot{2}$0$\dot{7}$ of £3 9*s.* $4\frac{1}{2}$*d.*
10. £·75 + 2·05*s.*; £·634375 + ·025 of 25*s.* + ·31$\dot{6}$ of 30*s.*
11. ·75 of 6*s.* 8*d.* − 1·84375 of 4*s.* + 3·9796 of 2*s.*
12. 2·8680$\dot{5}$ of 3*s.* + ·8$\dot{3}$ of 4*s.* − 1·8 of 5*s.*
13. 1·125 of £1 7*s.* + 44·045 of $11\frac{1}{4}$*d.* − ·0625 of 7*s.* 6*d.* + 1·025 of $3\frac{3}{4}$*d.*
14. ·175 ton + ·195 cwt. + ·145 qr. + ·15 lb.
15. ·573 in. + ·751 yd.; ·5 mile − ·375 fur.; ·16$\dot{3}$ of $2\frac{1}{2}$ miles.
16. 1·$\dot{2}$ of 3 A. 2 R. + ·1$\dot{3}$5 P. − 21·$\dot{9}$ yd. + 310·$\dot{1}$ ft.

DUODECIMALS.

Ex. 22 Find the product of—

1. 12 ft. 4 in. × 3 ft. 5 in.; 17 ft. 10 in. × 24 ft. 6 in.
2. 15 ft. 7 in. × 5 ft. 11 in.; 207 ft. 9 in. × 7 ft. 10 in.
3. 6 ft. 5 in. 4 pt. × 4 ft. 5 in.; 9 yd. 2 ft. 9 in. × 1 ft. 10 in.
4. 13 ft. 7 in. 11 pt. × 7 ft. 10 in. 4 pt.
5. 125 ft. 9 in. 8 pt. × 25 ft. 10 in. 3 pt.
6. 3 ft. 4 in. × 4 ft. 5 in. × 5 ft. 6 in.
7. 23 ft. 5 in. 6 pt. × 10 ft. 6 in. × 5 ft. 7 in.

Ex. 23. Find the quotient of—

1. 28 sq. ft. 66$\frac{2}{3}$ in. ÷ 4 ft. 5 in.
2. 80 sq. ft. 140 in. ÷ 5 ft. 6 in.
3. 11 sq. yd. 3 ft. 129 in. ÷ 2 ft. 9 in.
4. 84 sq. yd. 2 ft. 100 in. ÷ 11 ft. 11 in.
5. 683 sq. yd. 2 ft. 25 in. ÷ 52 yd. 10 in.
6. 180 sq. yd. 7 ft. 54 in. ÷ 7 ft. 10 in.

PRACTICE.

Ex. 24. Find the values of—

1. 59623 at 7$\frac{3}{4}$*d.*; 2654 at 8$\frac{1}{2}$*d.*; 38940 at 11$\frac{3}{4}$*d.*
2. 8765 at 14$\frac{1}{2}$*d.*; 18541 at 1*s.* 8$\frac{3}{4}$*d.*; 264 at 6*s.*
3. 2084 at 9*s.*; 20563 at 17*s.*; 43265 at 19*s.*
4. 2684 at 3*s.* 6*d.*; 6285 at 15*s.* 7*d.*; 1846 at 7*s.* 6$\frac{1}{2}$*d.*
5. 24503 at 13*s.* 7$\frac{3}{4}$*d.*; 265 at £3 15*s.*; 382 at £6 13*s.*
6. 123 at £5 13*s.* 6$\frac{1}{2}$*d.*; 356 at £7 15*s.* 7$\frac{3}{4}$*d.*
7. 366$\frac{5}{8}$ at £2 18*s.* 8*d.*; 1762$\frac{3}{5}$ at £3 5*s.* 7$\frac{1}{2}$*d.*
8. 3002$\frac{13}{22}$ at 18*s.* 3$\frac{3}{4}$*d.*; 6444$\frac{13}{16}$ at £5 7*s.* 5$\frac{1}{4}$*d.*

Ex. 25. Find the value of—

1. 31685 ft. 9 in. at 6*s.* 9$\frac{3}{4}$*d.* per ft.
2. 37 cwt. 2 qr. 14 lb. at £7 10*s.* 9*d.* per cwt.
3. 3 cwt. 1 qr. 7 lb. at 3*s.* 10*d.* per lb.
4. 9 oz. 15 dwt. 16 gr. at 2*s.* 11*d.* per oz.
5. 3 cwt. 2 qr. 21 lb. at £55 10*s.* per cwt.
6. 5 yd. 2 ft. 9 in. at 5*s.* 3$\frac{1}{2}$*d.* per ft.
7. 3 gal. 2 qt. 1 pt at 18*s.* 6*d.* per gallon.
8. 20 A. 3 R. 25 P. at £5 7*s.* 6$\frac{1}{2}$*d.* per acre.
9. 4 mo. 3 wk. 6 d. at 17*s.* 4*d.* per week.
10. 11 mo. 2 wk. 5 d. at £5 4*s.* 9*d.* per month.
11. 13 lb. 9 oz. 3 dwt. at £3 5*s.* per oz.
12. 17 cwt. 1 qr. 12 lb. at £1 19*s.* 8*d.* per cwt.
13. 356 A. 3 R. 39$\frac{1}{2}$ P. at £2 13*s.* 4*d.* per acre.
14. 45 A. 3 R. 20 P. at £111 11*s.* 4*d.* per acre.
15. 46 quarters 5 bush. at 58*s.* 8*d.* per quarter.

Ex. 25. Find the value of—

16. 7 quarters 3 bush. $3\frac{1}{2}$ pecks at 7*s.* 4*d.* per bushel.

17. 9600 rupees at 1*s.* 10*d.* each.

Ex. 26.

1. What is the dividend on £2045 15*s.* 9*d.* at 5*s.* $11\frac{1}{2}$*d.* in the £?

2. What is the tax on an income of 500 guineas at 7*d.* in the £1?

3. An officer's pay is 12*s.* 3*d.* per day; how much is that in a year?

RULE OF THREE DIRECT.

Ex. 27.

1. If 14*s.* will buy 8 lb. of tobacco; how much will £4 19*s.* $1\frac{1}{2}$*d.* buy at the same rate?

2. If the carriage of 3 cwt. 1 qr. 18 lb. of goods comes to £1 18*s.* $5\frac{1}{4}$*d.*; what will be the charge for carrying one ton the same distance?

3. If £$2\frac{1}{2}$ buy $3\frac{1}{3}$ gallons; how much will £$4\frac{1}{7}$ buy?

4. If $3\frac{1}{3}$ acres let for £$10\frac{1}{4}$; how much will $11\frac{1}{3}$ acres let for?

5. If $\frac{2}{3}$ of an estate be worth £220; what is the value of $\frac{3}{11}$ of it?

6. If 136 tons 13 cwt. of coals cost £182 4*s.*; find the cost of 370 tons 16 cwt.

7. If one bushel of malt cost 5*s.* 10*d.*; how much can I buy for £27 5*s.* 5*d.*?

8. If $34\frac{1}{2}$ yards of cloth cost £12 7*s.* $11\frac{5}{8}$*d.*; how many yards of this cloth can be bought for £3 19*s.* $0\frac{3}{4}$*d.*?

9. If $3\frac{5}{9}$ oz. avoird. cost 7*s.*; what will $10\frac{1}{7}$ lb. cost?

10. If an income of £5497 13*s.* 4*d.* yield a tax of £152 10*s.* 6*d.*; how much is this tax per £1?

11. If £$731\frac{3}{4}$ be the rent for 365 A. 3 R. 20 P.; what is the rent of 100 acres.

12. If £65 11*s.* buy 138 gallons of rum; find the cost of 475 gallons?

13. A bankrupt owes me £360 16*s.* 3*d.*, for which I get only £240 15*s.* 6*d.*; how much is this in the £?

14. If 1 oz. of tea cost $\cdot 458\dot{3}$*s.*; how much will £61 12*s.* buy?

15. How much per cent. is £62 of £75?

16. An article which cost 3*s.* 6*d.* is sold for 3*s.* $10\frac{1}{2}$*d.*; what is that per cent. profit?

Ex. 27.

17. If a tradesman gains 3*s.* 4½*d.* upon an article which cost him 15*s.* 9*d.*; how much does he gain per cent.?

18. A person sold 72 yards of cloth for £8 14*s.*, his profit being the cost of 11·52 yards; how much did he gain per cent.?

19. If 1 cwt. of an article cost £7; at what price per lb. must it be sold to gain 10 per cent.?

20. By selling a horse for £116 17*s.* a person lost 5 per cent.; what will be his gain or loss per cent. if he sell him for £132 4*s.* 6*d.*?

21. A tea-dealer buys a chest of tea containing 2 qr. 17 lb. at 3*s.* 1½*d.* per lb., and two chests, each containing 3 qr. 7 lb., at 3*s.* 5½*d.* per lb.; what will he gain per cent. by selling the mixture at 4*s.* per lb.?

22. A grocer buys coffee at £8 10*s.* per cwt., and chicory at £2 10*s.* per cwt., and mixes them in the proportion of 5 parts of chicory to 7 of coffee; at what rate must he sell the mixture so as to gain £16⅔ per cent. on his outlay?

23. If 13 tons 8 cwt. of goods cost £525; what will 3 cwt. 1 lb. 1½ oz. cost?

24. A shilling weighs 3 dwt. 15 gr., of which 3 parts out of 40 are alloy, and the rest pure silver. How much per cent. is there of alloy; and what is the weight of the pure silver?

25. If the pound weight of silver be coined into 66 shillings; what is the weight (avoirdupois) of half-a-crown?

26. If 1 lb. (troy) of standard gold be coined into £46 14*s.* 6*d.*; what is the weight of a sovereign?

RULE OF THREE INVERSE.

Ex. 28.

1. If a board be 8 inches broad; what must be its length to contain 12 square feet?

2. How many yards of matting ¾ yard wide, will be sufficient to cover a floor that is 15½ feet broad, and 27¾ feet long?

3. How many yards of paper ¾ yard wide, will hang a room 18 feet long, 15 broad, and 10 high?

4. If I lend a friend £100 for 12 months; for how long ought he to lend me £175 as an equivalent?

5. If 14 men can perform a piece of work in 17¾ days; in how many days can 35 men do the same work?

Ex. 28.

6. If 4 men *or* 6 women can do a piece of work in 20 days; how long will 3 men *and* 5 women take to finish it?

7. If a certain sum pay for the carriage of 100 lb. for 120 miles; how far ought 56 lb. to be carried for the same money?

8. A regiment of 1000 men are to have new coats; each coat is to contain $2\frac{1}{2}$ yards of cloth $1\frac{1}{4}$ yards wide, and to be lined with shalloon $\frac{3}{4}$ yard wide: how many yards of shalloon will be required?

9. If a garrison of 1500 men have provisions for 13 months; how long will their provisions last, if it be increased to 2200?

10. If 1000 men, marching 8 abreast, extend 325 paces; what will be their extent if they march 10 abreast?

11. If the step of a man be $2\frac{1}{3}$ feet, and that of a horse be $2\frac{3}{4}$ feet; how many horse-paces are equal to 50 man-paces?

12. How many roubles, each worth 3*s.* $4\frac{1}{2}$*d.*, are equal in value to 378 Napoleons at 15*s.* $9\frac{3}{4}$*d.* each?

13. If 2 cwt. 1 qr. 18 lb. of tea cost as much as 18 cwt. 9 lb. of sugar; how much sugar should be given in exchange for 1 lb. of tea?

DOUBLE RULE OF THREE.

COMPOUND PROPORTION.

Ex. 29.

1. If £120 be the wages of 6 men for 21 weeks; what will be the wages of 14 men for 46 weeks?

2. What must I pay for the work of 36 men for 7 months; when the wages of 50 men for 12 months amount to £1080?

3. If £$3\frac{1}{2}$ be the wages of 13 men for $7\frac{1}{2}$ days; what will be the wages of 20 men for $15\frac{1}{3}$ days?

4. If, with a capital of £1000, a man gains £100 in 5 months; in what time will he gain £49 10*s.* with a capital of £225?

5. If the carriage of 4 cwt. 3 qr. for 160 miles be £3 17*s.*; what will be the carriage of 11 cwt. 3 qr. 14 lb. for 100 miles?

6. If 11 cwt. can be carried 12 miles for a guinea; how far can 26 cwt. 23 lb. be carried for 5 guineas?

7. If a person can travel 142·2 miles in $4\frac{1}{2}$ days, each 10·164 hours long; how many days of 8·4 hours will he be in travelling 505·6 miles?

8. If 6 horses can plough $17\frac{1}{2}$ acres in 4 days; how many acres will 54 horses plough in $2\frac{1}{4}$ days?

Ex. 29.

9. If 10 horses consume 7 bushels 2 pecks in 7 days; in what time will 28 horses consume 3 qr. 6 bush., at the same rate?

10. If 48 men can perform a piece of work in 16 days of 9 hours each; in how many days of 12 hours each can 64 men accomplish the same?

11. How many days of 15 hours each would 60 men take to perform a piece of work in; when 45 men can do the same in 30 days of 12 hours each?

12. If 27 men can do a piece of work in 14 days, working 10 hours a day; how many hours a day must 24 boys work, in order to complete the same in 45 days: the work of a boy being half that of a man?

13. If 5 men and 7 boys can reap a field of corn of 125 acres in 15 days; in how many days will 10 men and 3 boys reap a field of corn of 75 acres, each boy's work being one-third of a man's?

14. If $13\frac{1}{2}$ ells of cloth, $\frac{3}{5}$ yard wide, cost $5\frac{1}{2}$ guineas; what will $33\frac{1}{4}$ yards, $\frac{2}{3}$ ell wide, come to?

15. If $\frac{1}{5}$ of a sheep be worth £$\frac{2}{3}$, and $\frac{3}{7}$ of a sheep be worth $\frac{1}{14}$ of an ox; how much must be given for 100 oxen?

16. If 12 oxen be worth 29 sheep, 15 sheep worth 25 hogs, 17 hogs worth 3 loads of wheat, and 8 loads of wheat worth 13 loads of barley; how many loads of barley must be given for 20 oxen?

17. If 12 of A count for 13 of B, 6 of B for 18 of C, and 13 of C for 2 of D; how many of A count for 100 of D?

18. If 48 men, working 8 hours a day for one week, can dig a trench 235 feet long, 40 wide, and 28 deep; in what time can 12 men, working 10 hours a day, form a railway cutting containing 156,060 cubic yards?

19. If the pound weight of standard gold, of 22 carats fine, be worth £46 14*s.* 6*d.*; find the values of the following gold coins, the weight and fineness (or number of parts of pure gold in 1000 parts of the coin) being as stated below:

(1). Mohur of Bengal, of weight 7 dwt. 23 gr. and fineness 993.
(2). Mohur of Bombay, of wt. 7 dwt. $10\frac{1}{2}$ gr. and fineness 953.
(3). Gold Rupee of Bombay, of wt. 7 dwt. 11 gr. and fineness 922.
(4). Gold Rupee of Madras, of wt. 7 dwt. 12 gr. and fineness 916.
(5). Star Pagoda of Madras, of wt. 2 dwt. $4\frac{1}{2}$ gr. and fineness 792.

20. If the pound weight of standard silver be worth 62*s.*, of which 222 parts in 240 are pure silver; find the values of the following silver coins:

(1). Sicca Rupee, of weight 7 dwt. 12 gr. and fineness 979.

Ex. 29.

(2). Arcott Rupee, of weight 7 dwt. 9 gr. and fineness 941.
(3). Bombay Rupee, of weight 7 dwt. 11 gr. and fineness 926.
(4). Baroch Rupee, of weight 7 dwt. 10 gr. and fineness 883.

SIMPLE INTEREST.

Ex. 30. Find the Simple Interest—

1. On £350, for 2 years, at 5 per cent.
2. On £530 17*s.* 6*d.*, for 11 years, at 5 per cent.
3. On £340 15*s.* 6*d.*, for 3 years, at 4 per cent.
4. On £235 14*s.* 4*d.*, for 3 years, at 5 per cent.
5. On £547 2*s.* 4*d.*, for $3\frac{1}{2}$ years, at 4 per cent.
6. On £300, for $3\frac{5}{12}$ years, at 2 per cent.

Ex. 31. Find the Amount—

1. Of £455, for $3\frac{1}{2}$ years, at 5 per cent.
2. Of £1643 7*s.* $5\frac{1}{2}$*d.*, for 4 years, at $3\frac{1}{2}$ per cent.
3. Of £575, for $8\frac{3}{4}$ years, at $3\frac{3}{8}$ per cent.
4. Of 1895 guineas, for $4\frac{3}{4}$ years, at $2\frac{3}{4}$ per cent.
5. Of £411 10*s.*, for $\frac{1}{4}$ year, at $4\frac{7}{8}$ per cent.
6. Of £2860 16*s.* $9\frac{3}{4}$*d.*, for $5\frac{1}{2}$ years, at $4\frac{1}{4}$ per cent.

Ex. 32. Find the Simple Interest and Amount—

1. Of £347 10*s.*, for 219 days, at 5 per cent.
2. Of £684 18*s.* 8*d.*, for 1 year 11 months, at $4\frac{1}{8}$ per cent.
3. Of £126 10*s.*, for 135 days, at $3\frac{1}{2}$ per cent.
4. Of 200 guineas, for 4 years, 7 months, 25 days, at $4\frac{1}{2}$ per cent.
5. Of £7500, from May 5 to Oct. 26, at $3\frac{1}{8}$ per cent.
6. Of £225 12*s.* 6*d.*, from Sept. 29 to Dec. 25, at $3\frac{1}{2}$ per cent.

Ex. 33. In what time, at Simple Interest,—

1. Will £150 amount to £165 15*s.*, at 3 per cent.?
2. Will £142 10*s.* amount to £227 5*s.* 9*d.*, at $3\frac{1}{2}$ per cent.?
3. Will £1275 amount to £1549 11*s.*, at $3\frac{4}{5}$ per cent.?
4. Will £100 amount to £1000, at 5 per cent.?
5. Will a given sum double itself, at $3\frac{1}{2}$ per cent.?
6. Will a given sum treble itself, at $4\frac{1}{4}$ per cent.?

Ex. 34. At what rate per cent. per annum, Simple Interest,—

1. Will £300 amount to £350 in 7 years?
2. Will £157 15*s*. 4*d*. amount to £295 16*s*. 3*d*. in 25 years?
3. Will £936 13*s*. 4*d*. amount to £1157 7*s*. $4\frac{1}{2}$*d*. in $4\frac{7}{8}$ years?
4. Will a given sum double itself in 30 years?
5. Will £200, in 146 days, produce £4 16*s*.?

Ex. 35. What sum or principal put out at Simple Interest—

1. Will amount to £111, in 5 years, at 4 per cent.?
2. Will amount to £600, in 6 years, at 4 per cent.?
3. Will amount to £105 6*s*. $0\frac{1}{2}$*d*., in $3\frac{1}{2}$ years, at $4\frac{1}{4}$ per cent.?
4. Will produce £455 per annum, at $3\frac{1}{4}$ per cent.?
5. Will produce £56 14*s*. interest, in $2\frac{1}{4}$ years, at $4\frac{1}{2}$ per cent.?

COMPOUND INTEREST.

Ex. 36. Find the Compound Interest—

1. Of £256 10*s*., in 4 years, at 5 per cent.
2. Of £690, in 3 years, at $4\frac{1}{2}$ per cent.
3. Of £317 16*s*., in 5 years, at 3 per cent.
4. Of £760 10*s*., in 4 years, at 4 per cent.

Ex. 37. Find the Amount, Compound Interest,—

1. Of £430, in 3 years, at 4 per cent.
2. Of £275 15*s*., in $4\frac{1}{2}$ years, at 4 per cent.
3. Of £845 17*s*. 10*d*., in 5 years, at 3 per cent.
4. Of £2643 13*s*. 8*d*., in $2\frac{1}{2}$ years, at 3 per cent.
5. Of £244 17*s*. $6\frac{1}{2}$*d*., in $3\frac{3}{4}$ years, at $2\frac{1}{2}$ per cent.
6. Of £420 15*s*., in 4 years, at 3 per cent., payable half-yearly.

Ex. 38. Find the difference between the Simple and Compound Interest—

1. Of £256, in 3 years, at $4\frac{1}{2}$ per cent.
2. Of £13333 6*s*. 8*d*., in 5 years, at 5 per cent.
3. Of $100\frac{1}{2}$ guineas, in 3 years, at $4\frac{1}{2}$ per cent.
4. Of £38 10*s*. 6*d*., in $2\frac{1}{2}$ years, at 5 per cent.

Ex. 39.

1. What sum, at 4 per cent. per annum, Compound Interest, will amount in 2 years to £405 12*s*.?

Ex. 39.

2. What principal, at 3 per cent. per annum, Compound Interest, will amount in 6 years to £597 0*s*. 6$\frac{1}{4}$*d*.?

DISCOUNT.

Ex. 40. Find the Discount—

1. On £1000, due 4 years hence, at 5 per cent.
2. On £256 7*s*. 6*d*., due 3 years hence, at 3 per cent.
3. On £1380 7*s*. 6*d*., due 9 months hence, at 3 per cent.
4. On £275 6*s*. 8*d*., due 18 months hence, at 4$\frac{1}{2}$ per cent.
5. On £1062 10*s*., due 1 year 4 months hence, at 3$\frac{3}{4}$ per cent.
6. On £55, due 146 days hence, at 4$\frac{3}{4}$ per cent.
7. On a bill of £131 3*s*. 6*d*., drawn August 1, at 4 months, and discounted September 12, at 5 per cent.

Ex. 41. Find the Present Worth—

1. Of £903 14*s*., due 2$\frac{1}{8}$ years hence, at 3$\frac{1}{2}$ per cent. per ann.
2. Of £813 9*s*., due 1$\frac{1}{3}$ years hence, at 4$\frac{3}{4}$ per cent. —
3. Of £676 13*s*. 4*d*., due 6 months hence, at 3 per cent. —
4. Of £324 16*s*. 7$\frac{1}{2}$*d*., due 2$\frac{3}{4}$ years hence, at 3$\frac{1}{2}$ per cent. —
5. Of 800 guineas, due 20 years hence, at 5$\frac{1}{2}$ per cent. —
6. Of £2197, due in 3 years, at 4 per cent., Compound Interest.

STOCKS.

Ex. 42. Find the quantity of Stock purchased by investing—

1. £500 in the 3$\frac{1}{2}$ per cents at 94.
2. £4311 8*s*. 9*d*. in the 3$\frac{1}{2}$ per cents at 85$\frac{3}{8}$.
3. 400 guineas in the 4 per cents at 94.
4. £1606 in the 4 per cents at 100$\frac{1}{4}$.
5. £588 5*s*. in the 3$\frac{1}{4}$ per cents at 90$\frac{1}{2}$.
6. £3097 in the 5 per cents at 105$\frac{3}{8}$, brokerage $\frac{1}{8}$ per cent.

Ex. 43. Find the value in Sterling Money—

1. Of £1000, 4 per cent. stock at 82$\frac{1}{8}$.
2. Of £439 12*s*. 6*d*., 3$\frac{1}{2}$ per cent. stock at 92$\frac{7}{8}$.
3. Of £10,000, 3 per cent. stock at 93$\frac{3}{4}$.
4. Of £4000, 3$\frac{1}{4}$ per cent. stock at 97$\frac{1}{8}$.
5. Of £2153 10*s*., Bank stock at 188$\frac{7}{8}$, brokerage $\frac{1}{8}$ per cent.

Ex. 44. Find the yearly income arising from the investment—

1. Of £2000 in the 3 per cents at $88\frac{1}{2}$.

2. Of £1047 1*s.* 8*d.* in the 3 per cents at $89\frac{3}{4}$.

3. Of £3995 in the $3\frac{1}{2}$ per cents at $99\frac{7}{8}$.

4. Of £3003 in the Dutch $2\frac{1}{2}$ per cents at $49\frac{1}{2}$.

5. Of 5000 guineas in the $3\frac{1}{4}$ per cents at $102\frac{3}{8}$, brokerage $\frac{1}{8}$ per cent.

Ex. 45.

1. What interest per cent. is derived from investing money in the $3\frac{1}{2}$ per cents at $101\frac{1}{2}$? and in the 3 per cents at $93\frac{3}{4}$?

2. If a person receive $4\frac{1}{2}$ per cent. interest on his capital by investing it in the $3\frac{1}{2}$ per cents; what is the price of the stock, and how much stock can be purchased for £1200?

3. A person transfers £5000 stock from the $3\frac{1}{4}$ per cents at 98 to the 3 per cents at 94; how much of the latter stock will he hold, and what will be the difference in his income?

4. What would be the difference in annual income from investing £15000 in the 5 per cents at $110\frac{1}{2}$, and in the $3\frac{1}{2}$ per cents at $92\frac{7}{8}$?

5. A person invests £18150 in the 3 per cents at $90\frac{3}{4}$, and on the stock rising to 91, transfers it to the $3\frac{1}{2}$ per cents at $97\frac{1}{2}$; what increase in his annual income is thereby effected?

6. If £1000 of 3 per cent. stock at 72 be transferred to the 4 per cents at 90; find the alteration of income.

SINGLE FELLOWSHIP.

Ex. 46.

1. Four persons, A, B, C and D, rent a pasture for £50; A put in 7 cattle; B, 8; C, 9; and D, 10: how much should each person pay for his share?

2. A tax of £530 is to be raised from 3 towns, the numbers of inhabitants of which respectively are 2500, 3000, and 4200. How much should each pay?

3. A ship is wrecked, whose value is £1500, of which £800 belonged to A, £400 to B, £200 to C, and £100 to D. The insurance office pays them £1250 of their loss: how much of this should each receive?

4. Three persons whose estates are worth respectively £1000, £755, and £645 a-year, buy 100 railway shares among them, each buying a number proportional to his estate. How many shares does each buy?

DOUBLE FELLOWSHIP.

Ex. 47.

1. Four merchants, A, B, C and D, trade together. A's stock of £300 was in trade 12 months; B's stock of £330 for 10 months; C's of £375 for 8 months; and D's of £395 for 6 months. The whole gain being £723; how much ought each to receive?

2. A company, consisting of 1 captain, 2 lieutenants, 6 sergeants, 10 corporals, and 50 men, storm a fort, and find there a treasure worth £1600, which is divided among them according to their pay and the time of their service. The captain has been in the service for 5 years, and is paid 11*s.* 7*d.* a day; the lieutenants have served $3\frac{1}{2}$ years, and are paid 6*s.* 6*d.* a day; the sergeants have served 7 years, and are paid 2*s.* 6*d.* a day; the corporals have served 4 years, and are paid 1*s.* 6*d.* a day; and the men have served 2 years, and are paid 1*s.* a day: what portion of the prize should each receive?

EXTRACTION OF ROOTS.

Ex. 48. Find the square roots of—

1. 2601; 7225; 9801; 47089; 138384.
2. 390625; 553536; 5764801; 43046721.
3. 2; 150; 1053; 3000; 7658.
4. 75·347; ·4325; 876·535; ·000729.
5. 36·00000625; 1195·50669121; ·0900375.
6. $5\cdot\dot{3}$; $1\cdot\dot{7}$; $6\cdot24\dot{9}$; $1788\cdot5\dot{7}$.
7. $\frac{225}{324}$; $27\frac{9}{16}$; $29\frac{4}{25}$; $275\frac{1}{144}$.
8. $\frac{7}{9}$; $\frac{45}{94}$; $\frac{448}{507}$; $15\frac{5}{8}$; $8\frac{3}{7}$.
9. $\frac{1}{17}$; $\frac{5}{49}$; $76\frac{14}{17}$; $\frac{5\cdot04}{\cdot012}$.

Ex. 49. Find the cube roots of—

1. 97336; 405224; 941192; 2985984.
2. 134217728; 273·359449.
3. ·7854; 1·092727; ·001533.
4. ·007077888; ·000057464.
5. $\frac{324}{150}$; $5\frac{104}{105}$; $\frac{5}{9}$; $\frac{23}{24}$.

Ex. 50. Find the fourth roots of—

1. 6561; 1679616; 2266·7121.
2. 6724; 1·2544; ·96059601.

Ex. 51.

1. Find the sixth roots of 531441; 262144; 19683.
2. Divide the cube root of $\frac{2515456}{32768}$ by the fourth root of 8·3521.
3. Add together the cube roots of ·059319 and 4·173281; and multiply the sum by the square root of $105\frac{1}{16}$.

ALGEBRA.

Ex. 1. Find the value—

1. Of $3a^2+\frac{2ab^2}{c}-\frac{c^3}{b^2}$; $\frac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$; when $a=4$, $b=\frac{1}{2}$, $c=1$.

2. Of $x^2+(x^2-42x+89)^{\frac{1}{2}}$, when $x=2$.

3. Of $\frac{a}{b}-\left(\frac{1+a}{1-b}\right)^{\frac{1}{2}}$, when $a=\frac{1}{4}$, $b=\frac{1}{5}$.

4. Of $(4x^4-12x^3+12x^2-4)^{\frac{1}{3}}$, when $x=2$.

MULTIPLICATION.

Ex. 2.

1. Multiply ab^4 by a^2bc; $2mx^3y$ by $-3nxy^5$; $-h^2tx^3$ by $-k^3t^2x^7$.
2. Multiply $3x^3-2x^2y+2xy^2-y^3$ by $4x^2y^2$.
3. Multiply $3x-y$ by $2x+5y$; and $4x^2+3xy$ by $x-3y$.
4. Multiply $12x^3-8x^2y+15xy^2-10y^3$ by $3x+2y$.
5. Multiply a^2+ax+x^2 by a^2-ax+x^2.
6. Multiply $x^2+2xy-3y^2$ by $x^2-5xy+4y^2$.
7. Multiply x^3-ax^2+bx-c by x^2-tx+s.
8. Multiply $9x^2+3xy+y^2-6x+2y+4$ by $3x-y+2$.
9. Multiply $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.
10. Multiply $\frac{5}{2}x^2+3ax-\frac{7}{3}a^2$ by $2x^2-ax-\frac{1}{2}a^2$.
11. Multiply $\frac{3}{2}h^3-5h^2+\frac{1}{4}h+9$ by $\frac{1}{2}h^2-h+3$.
12. Multiply $x^4+(2a^2-b^2)x^2+a^4+a^2b^2$ by $x^2-a^2-b^2$.
13. Multiply $a^m+b^p-2c^n$ by $2a^m-3b$.
14. Multiply $x^{(m-1)n}-y^{(n-1)m}$ by x^n-y^m.
15. Multiply together $x-3$, $x+4$, $x-5$, and $x+6$.
16. Multiply together $2x-1$, $x^2+\frac{1}{4}$ and $2x+1$.
17. Multiply together $x+3$, $x-7$, $x+4$ and $x-10$.
18. Multiply together $3x-4y$, $x-2y$, $x+2y$, and $3x+4y$.

DIVISION.

Ex. 3.

1. Divide $75a^5b^2$ by $5a^2b^2$; and $-147x^3y^7z^4$ by $-7xy^2z$.
2. Divide $9x^2y^2-15x^3y^4z+6x^5y^3z^2$ by $3x^2y$.
3. Divide $6x^2+5xy-4y^2$ by $3x+4y$.
4. Divide $x^3-40x-63$ by $x-7$.
5. Divide $3h^5+16h^4k-33h^3k^2+14h^2k^3$ by h^2+7hk.
6. Divide a^5-243 by $a-3$; and $x^{15}+y^{10}$ by x^3+y^2.
7. Divide $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.
8. Divide $1-6x^5+5x^6$ by $1-2x+x^2$.
9. Divide $p^2+pq+2pr-2q^2+7qr-3r^2$ by $p-q+3r$.
10. Divide $x^3-8y^3+125z^3+30xyz$ by $x-2y+5z$.
11. Divide $x^6-140x^4+1050x^3-3101x^2+3990x-1800$ by $x^3-12x^2+47x-60$.
12. Divide $x^4-\frac{19}{6}x^2y^2+\frac{1}{3}xy^3+\frac{1}{6}y^4$ by $x^2+2xy+\frac{1}{3}y^2$.
13. Divide $-\frac{5}{9}a^2+\frac{11}{3}ab-\frac{10}{3}ac+\frac{15}{4}b^2+25bc$ by $-\frac{2}{3}a+5b$.
14. Divide $abx^3+(ac-bd)x^2-(af+cd)x+df$ by $ax-d$.
15. Divide $(x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)$ by $(x-1)a^2-(x-1)a+3$.
16. Divide $-2x^5y^{-8}+17x^6y^{-4}-5x^7-24x^8y^4$ by $-x^2y^{-5}+7x^3y^{-1}+8x^4y^3$.
17. Divide $1+2x$ by $1-3x$, to 5 terms in the quotient.
18. Divide $1-3x-2x^2$ by $1-4x$, to 6 terms in the quotient.
19. Divide $x^{pq}-1$ by x^p-1; and write down the last three terms of the quotient, when q denotes an integer.
20. Divide $a^nb-ab^n-a^nc+ac^n+b^nc-bc^n$ by $(a-b)(a-c)$.

GREATEST COMMON MEASURE.

Ex. 4. Find the greatest common measure—

1. Of $6a^2xy^3$ and $8abx^2yz$; also of $9x^3z^2-21xy^4z$ and $-3axz^4$.
2. Of $5a^2x^2-15axy$ and $10ax^3+35a^2xy^2$.
3. Of $4a^2-b^2$ and $4a^2+2ab-2b^2$.
4. Of $a^2-ab-12b^2$ and $a^2+5ab+6b^2$.
5. Of $5x^2-2x-3$ and $5x^2-11x+6$.
6. Of $9x^2-4$ and $9x^2-15x-14$.
7. Of $2x^4+11x^3-13x^2-99x-45$ and $2x^3-7x^2-46x-21$.

Ex. 4. Find the greatest common measure—

8. Of $x^4+x^3y-9x^2y^2+11xy^3-4y^4$ and $x^4-x^3y-3x^2y^2+5xy^3-2y^4$.
9. Of $6x^4-25a^2x^2-9a^4$ and $3x^3-15ax^2+a^2x-5a^3$.
10. Of $21x^3-26x^2+8x$ and $6x^2-x-2$.
11. Of $5x^3+2x^2-15x-6$ and $-7x^3+4x^2+21x-12$.
12. Of $6a^4-a^3b-3a^2b^2+3ab^3-b^4$ and $9a^4-3a^3b-2a^2b^2+3ab^3-b^4$.
13. Of $6a^4x^3-10a^2x^4y-9a^3x^2y^2+15ax^3y^3$ and $10a^4xy^2-15a^3y^4+8a^2x^2y^3-12axy^5$.
14. Of $ab+2a^2-3b^2+4bc+ac-c^2$ and $2a^2-9ac-5ab+4c^2-8bc-12b^2$.
15. Of $3x^2+(4a-2b)x-2ab+a^2$ and $x^3+(2a-b)x^2-(2ab-a^2)x-a^2b$.
16. Of $e^{2x}a^3+e^{2x}-a^3-1$ and $e^{2x}a^2+2e^xa^2-e^{2x}-2e^x+a^2-1$.
17. Of $3x^3-7x^2y+5xy^2-y^3$, $x^2y+3xy^2-3x^3-y^3$, and $3x^3+5x^2y+xy^2-y^3$.

LEAST COMMON MULTIPLE.

Ex. 5. Find the least common multiple—

1. Of $8a^3bx^2$ and $14ab^2x^3$; of $3xy^2$ and $5x^2y^2-4x^3y$.
2. Of $6a^2$, $9ax^3$ and $24x^5$; of $32x^2y^2$, $40ax^5y$ and $5a^2x(x-y)$
3. Of $3x+6y$ and $2x^2-8y^2$; of a^3+x^3 and a^2-x^2.
4. Of $21x^2-26x+8$ and $7x^3-4x^2-21x+12$.
5. Of $4(a^2+ax)$, $12(ax^2-x^3)$, and $18\ (a^2-x^2)$.
6. Of x^2-1, x^2+2x-3, and x^3-7x^2+6x.
7. Of x^2-y^2, $3(x-y)^2$, and $12(x^3+y^3)$.
8. Of x^2-1, x^2+1, $(x-1)^2$, $(x+1)^2$, x^3-1, and x^3+1.

FRACTIONS.

Ex. 6. Reduce to their lowest terms—

1. $\dfrac{15ab^3-3a^2bc^2}{5abc}$; $\dfrac{14a^2-7ab}{10ax-5bx}$; $\dfrac{11x^2+11ax}{x^2-a^2}$.
2. $\dfrac{x^2-4x+3}{x^2-2x-3}$; $\dfrac{6x^2-5x-6}{4x^3-9x}$; $\dfrac{x^3-4x^2+5}{x^3+1}$.
3. $\dfrac{x^3-39x+70}{x^2-3x-70}$; $\dfrac{x^3-19x^2+119x-245}{3x^2-38x+119}$.

Ex. 6. Reduce to their lowest terms—

4. $\dfrac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}$; $\dfrac{2x^3+8x^2y+16xy^2+16y^3}{8x^2+4xy-24y^2}$.

5. $\dfrac{ac+by+ay+bc}{af+2bx+2ax+bf}$; $\dfrac{ac+9bc-5c^2}{2adf+18bdf-10cdf}$; $\dfrac{acx^2+(ad-bc)x-bd}{a^2x^2-b^2}$.

6. $\dfrac{4a^3cx-4a^3dx+24a^2bcx-24a^2bdx+36ab^2cx-36ab^2dx}{7abcx^3-7abdx^3+7ac^2x^3-7acdx^3+21b^2cx^3+21bc^2x^3-21b^2dx^3-21bcdx^3}$.

Ex. 7. Show that—

1. $\dfrac{30x^3+11x^2y-38xy^2-40y^3}{15x^2-17xy-4y^2}=2x+3y+\dfrac{7y^2}{5x+y}$.

2. $\dfrac{a^3-b^3}{(a-b)^2}=a+2b+\dfrac{3b^2}{a-b}$.

3. $\dfrac{60x^3-17x^2-4x+1}{5x^2+9x-2}=12x-25+\dfrac{49}{x+2}$.

4. $\dfrac{x^4-9x^2y^2+64y^4}{x^3-6x^2y+13xy^2-8y^3}=x+6y+\dfrac{14y^2}{x-y}$.

5. $a^2-ab+b^2-\dfrac{2b^3}{a+b}=\dfrac{a^3-b^3}{a+b}$; $1+x+x^2+\dfrac{x^3}{1-x}=\dfrac{1}{1-x}$.

6. $(a-b)^2+\dfrac{6a^2b+2b^3}{a-b}=\dfrac{(a+b)^3}{a-b}$.

7. $a^2-6ab+17b^2-\dfrac{16b^3(2a+b)}{(a+b)^2}=\dfrac{(a-b)^4}{(a+b)^2}$.

8. $1+\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{(a+b+c)(a+b-c)}{2ab}$.

9. $1-\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}=\dfrac{(a+c+d-b)(b+c+d-a)}{2(ab+cd)}$.

Ex. 8. Find the value—

1. Of $\dfrac{ax}{b}+\dfrac{by}{c}$; $\dfrac{x}{x+y}+\dfrac{y}{x-y}$; $\dfrac{a+b}{a-b}-\dfrac{2ab}{a^2-b^2}$.

2. Of $\dfrac{1}{2(x-1)}-\dfrac{1}{2(x+1)}-\dfrac{1}{x^2}$; $\dfrac{p-q}{pq}+\dfrac{r-p}{pr}+\dfrac{q-r}{qr}$.

3. Of $\dfrac{1}{4(1+x)}+\dfrac{1}{4(1-x)}+\dfrac{1}{2(1+x^2)}$; $\dfrac{1}{x^2}+\dfrac{x-1}{x^2+1}-\dfrac{1}{(x^2+1)^2}$.

4. Of $\dfrac{1}{2x+y}+\dfrac{1}{2x-y}-\dfrac{3x}{4x^2-y^2}$; $\dfrac{x+y}{y}-\dfrac{x}{x+y}-\dfrac{x^3-x^2y}{x^2y-y^3}$.

5. Of $\dfrac{3}{4(1-x)^2}+\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}-\dfrac{1-x}{4(1+x^2)}$.

Ex. 8. Find the value—

6. Of $\dfrac{4x+13}{25(x+2)^2}-\dfrac{4x-3}{25(x^2+1)}$.

7. Of $\dfrac{a}{c}-\dfrac{(ad-bc)x}{c(c+dx)}$; $\qquad \dfrac{a}{b}-\dfrac{(a^2-b^2)x}{b^2}+\dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}$.

8. Of $\dfrac{1}{1-x}-\dfrac{1}{(1-x)^2}+\dfrac{1}{(1-x)^3}-\dfrac{1}{(1-x)^4}$.

9. Of $\dfrac{1}{x^3}+\dfrac{1}{x^2}-\dfrac{1}{x}+\dfrac{x-1}{x^2+1}-\dfrac{1}{(x^2+1)^2}$.

10. Of $\dfrac{1}{(x+1)(x+2)}-\dfrac{1}{(x+1)(x+2)(x+3)}$.

11. Of $\dfrac{1}{(x+1)(x+2)}-\dfrac{3}{(x+1)(x+2)(x+3)}$.

12. Of $\dfrac{1+2x}{(3-x)(1+x)}+\dfrac{7}{(2+x)(1-3x)}+\dfrac{x}{(1+x)(2+x)}$.

13. Of $\dfrac{1-x}{1+x}+\dfrac{(1-x)(1-x^2)}{(1+x)(1+x^2)}+\dfrac{(1-x)(1-x^2)(1-x^3)}{(1+x)(1+x^2)(1+x^3)}$.

14. Of $\dfrac{1-x}{1+x}+\dfrac{1-x-x^2}{1+x+x^2}+\dfrac{1-x-x^2-x^3}{1+x+x^2+x^3}$.

15. Of $\dfrac{a+c}{(a-b)(x-a)}-\dfrac{b+c}{(a-b)(x-b)}$.

16. Of $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}+\dfrac{1}{c(c-a)(c-b)}$.

17. Of $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-a)(b-c)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}$.

18. Of $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-a)(b-c)(x-b)}+\dfrac{c}{(c-a)(c-b)(x-c)}$.

19. Of $\dfrac{a^2}{(a-b)(a-c)(x-a)}+\dfrac{b^2}{(b-a)(b-c)(x-b)}+\dfrac{c^2}{(c-a)(c-b)(x-c)}$.

20. Of $\dfrac{a^2+ha+k}{(a-b)(a-c)(x-a)}+\dfrac{b^2+hb+k}{(b-a)(b-c)(x-b)}+\dfrac{c^2+hc+k}{(c-a)(c-b)(x-c)}$.

21. Of $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$.

Ex. 9. Required in their simplest forms, the products—

1. Of $\dfrac{3x}{4}\times\dfrac{4x}{3y}$; $\qquad \dfrac{8a^2b}{c}\times\dfrac{c^2d}{8a^3}$; $\qquad \dfrac{4ax}{cy}\times\dfrac{cxy-y^2}{6x^2+6xy}$.

Ex. 9. Required in their simplest forms, the products—

2. Of $\left(a-\frac{x^2}{a}\right)\times\left(\frac{a}{x}+\frac{x}{a}\right)$; $\left(1+\frac{3x}{a-x}\right)\times\left(\frac{a-x}{a+2x}\right)^2$.

3. Of $\left(a-\frac{x^2}{a-x}\right)\times\left(x-\frac{x^2}{a+x}\right)$; $\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3}\times\frac{a^2-ax+x^2}{a+x}$.

4. Of $\frac{x^2-9x+20}{x^2-6x}\times\frac{x^2-13x+42}{x^2-5x}$; $\frac{x^2+3x+2}{x^2+2x+1}\times\frac{x^2+5x+4}{x^2+7x+12}$.

5. Of $\left(\frac{4a}{3x}+\frac{3x}{2b}\right)\times\left(\frac{2b}{3x}+\frac{3x}{4a}\right)$; $\left(\frac{a^2}{x^2}-\frac{ab}{2xy}+\frac{b^2}{y^2}\right)\times\left(\frac{3a^2}{x^2}-\frac{2ab}{5xy}+\frac{b^2}{y^2}\right)$.

6. Of $\left(\frac{a^2}{b^3}+\frac{7c^2}{2a^4b^3}+\frac{2c^3d^4}{b^5}\right)\times\left(\frac{a^2}{b^3}-\frac{2c^3d^4}{b^5}-\frac{7c^2}{2a^4b^3}\right)$.

7. Of $\frac{pr+(pq+qr)x+q^2x^2}{p-qx}\times\frac{ps+(pt-qs)x-qtx^2}{p+qx}$.

8. Of $\left(\frac{x}{a}-\frac{y}{b}\right)\frac{z}{c}+\left(\frac{x}{a}-\frac{z}{c}\right)\frac{y}{b}+\left(\frac{y}{b}-\frac{z}{c}\right)\frac{x}{a}$.

9. Of $\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{c}{d}+\frac{d}{c}\right)-\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{c}{d}-\frac{d}{c}\right)$.

Ex. 10. Required in their simplest forms, the quotients—

1. Of $\frac{5a^2}{b}\div\frac{3c^2}{5d}$; $\frac{a}{b}\div\frac{a-x}{c}$; $\frac{a^2+b^2}{a^2-b^2}\div\frac{a-b}{a+b}$.

2. Of $\left(4x-\frac{3y^2}{x}\right)\div\left(9x+\frac{8y^3}{x^2}\right)$; $\left(\frac{a}{b}+\frac{c}{d}\right)\div\left(\frac{e}{f}-\frac{g}{h}\right)$.

3. Of $\frac{x^3+y^3}{x^2-y^2}\div\frac{x^2-xy+y^2}{x-y}$; $\frac{a^4-x^4}{a^2-2ax+x^2}\div\frac{a^2x+x^3}{a^3-x^3}$.

4. Of $\left(\frac{1}{1+x}+\frac{x}{1-x}\right)\div\left(\frac{1}{1-x}-\frac{x}{1+x}\right)$.

5. Of $\left(\frac{x}{1+x}+\frac{1-x}{x}\right)\div\left(\frac{x}{1+x}-\frac{1-x}{x}\right)$.

6. Of $\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$.

7. Of $\left(\frac{a}{b}-\frac{c}{d}+\frac{e}{f}\right)\div\left(\frac{s}{t}+\frac{x}{y}+\frac{z}{w}\right)$.

8. Of $\left(\frac{a^2}{bc}-\frac{2a}{d}+\frac{ac}{be}+\frac{bc}{d^2}-\frac{c^2}{de}\right)\div\left(\frac{a}{b}-\frac{c}{d}\right)$.

9. Of $\left(\frac{a^2x^3}{bd}+\frac{abx^2}{c^2d}-\frac{acx^2}{d^2}-\frac{b^2x}{cd^2}+\frac{a^2x}{bc}-\frac{a}{d}\right)\div\left(\frac{ax}{c}-\frac{b}{d}\right)$.

Ex. 10. Required in its simplest form, the quotient—

10. Of $\dfrac{a^2+(2ac-b^2)x^2+c^2x^4}{a^2+2abx+(2ac+b^2)x^2+2bcx^3+c^2x^4} \div \dfrac{a^2+(ac-b^2)x^2+bcx^3}{a^2+(ac-b^2)x^2-bcx^3}$.

Ex. 11. Find the value—

1. Of $\left(a+\dfrac{b-a}{1+ba}\right)\times\dfrac{a}{b}\div\left(1-a\dfrac{b-a}{1+ba}\right)$.

2. Of $\dfrac{1}{1+x^{m-n}+x^{m-p}}+\dfrac{1}{1+x^{n-m}+x^{n-p}}+\dfrac{1}{1+x^{p-m}+x^{p-n}}$.

3. Of $\dfrac{1}{\dfrac{1}{x+3}+\dfrac{1}{x-5}+\dfrac{1}{x+7}}$; $\dfrac{x}{1+\dfrac{x}{1+x+\dfrac{x}{1+x+x^2}}}$.

4. Of $\dfrac{3\frac{1}{4}-\frac{1}{3}(x-2)}{1\frac{1}{12}+(x-\frac{3}{2})}$; $\dfrac{\dfrac{a+bx}{a-bx}+\dfrac{b+ax}{b-ax}}{\dfrac{a+bx}{a-bx}-\dfrac{b+ax}{b-ax}}$.

Ex. 12. Find the value—

1. Of $\dfrac{x^3-1}{x^3-4x^2+4x-1}$; when $x=1$.

2. Of $\dfrac{x^2+2x-35}{x^2-6x+5}$; $\dfrac{2x^2-11x+5}{3x^3-13x^2-10x}$; when $x=5$ in each case.

3. Of $\dfrac{xe^{2x}+1-e^{2x}-x}{e^{2x}-1}$; when $x=0$.

4. Of $\dfrac{x^5+ax^4-a^4x-a^5}{x^4+2ax^3-2a^2x^2-2a^3x+a^4}$; when $x=a$.

INVOLUTION AND EVOLUTION.

Ex. 13. Find by involution the value—

1. Of $(3ax^2z^3)^2$; $(-5a^2bx^3)^3$; $\left(-\dfrac{4p^3q^5}{3r^2}\right)^4$.

2. Of $(x-5y)^2$; $(3x+2y)^2$; $(x^2-3y^2)^2$.

3. Of $(1-2x+3x^2)^2$; $(x^3-2x^2+x-2)^2$.

4. Of $(2x+5y-3z)^2$; $(3ax-2by+cz)^2$.

5. Of $\left(\dfrac{x}{y}-\dfrac{3y}{x}\right)^2$; $\left(\dfrac{x}{2}-\dfrac{y}{3}+\dfrac{z}{4}\right)^2$; $\left(\dfrac{a^2}{x}-\dfrac{2b^2}{y}+\dfrac{3c^2}{z}\right)^2$

6. Of $(2x^2-3y^2)^3$; $\left(\dfrac{x}{a}+\dfrac{a}{x}\right)^3$; $(e^x-e^{-x})^3$.

Ex. 13. Find by involution the value—

7. Of $(1-2x+3x^2)^3$; $\left(\frac{x^2}{y^2}-2+\frac{y^2}{x^2}\right)^3$.

8. Of $(a+c-b)^3+(a+b-c)^3+(b+c-a)^3+24abc$.

9. Of $(2x-1)^4$; $(x-3y)^5$; $\left(x^2-\frac{a}{c}\right)^6$.

10. Of $(px\mp qz)^5$; $\left(x^2-1+\frac{1}{x^2}\right)^4$.

Ex. 14. Extract the square root—

1. Of $9a^6x^4y^8$; $\frac{64x^2y^{10}}{25z^6}$; $\frac{144a^{12}b^4}{81c^6d^{14}}$.

2. Of $9a^2+6ab+b^2$; $16x^2-40xy+25y^2$.

3. Of $4x^2z^2+12xyz+9y^2$; $36a^4x^2y^2-\frac{ax}{z}+\frac{1}{144a^2y^2z^2}$.

4. Of $1+2x+7x^2+6x^3+9x^4$; $4x^4-12x^3+25x^2-24x+16$.

5. Of $4a^2-12ab+9b^2+4ac-6bc+c^2$.

6. Of $16b^4-16ab^2x+16b^2x^2+4a^2x^2-8ax^3+4x^4$.

7. Of $9x^2-6xy+30xz+6xt+y^2-10yz-2yt+25z^2+10zt+t^2$.

8. Of $9x^2-30ax-3a^2x+25a^2+5a^3+\frac{a^4}{4}$.

9. Of $x^2+\frac{2ax}{3}-bx+\frac{a^2}{9}-\frac{ab}{3}+\frac{b^2}{4}$.

10. Of $\frac{x^2}{25}+\frac{xy}{15}+\frac{y^2}{36}-\frac{2xz}{5}-\frac{yz}{3}+z^2$.

11. Of $\frac{a^2}{x^2}+\frac{ab}{xy}+\frac{b^2}{4y^2}-\frac{6ac}{xz}-\frac{3bc}{yz}+\frac{9c^2}{z^2}$.

12. Of $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-1\frac{3}{4}$.

13. Of $25\frac{3}{7}-\frac{20x}{7y}+\frac{9y^2}{16x^2}-\frac{15y}{2x}+\frac{4x^2}{49y^2}$.

14. Of $\frac{x^4}{4y^4}+\frac{4y^4}{x^4}+\frac{x^2}{y^2}+\frac{4y^2}{x^2}+3$.

15. Of $x^6-4x^5y+10x^4y^2-20x^3y^3+25x^2y^4-24xy^5+16y^6$.

16. Of $1-x^2$; x^2+a^2; x^2-2ax; each to 5 terms.

Ex. 15. Extract the cube root—

1. Of $27x^6y^{15}$; $\frac{343x^3z^6}{64y^{12}}$; $-\frac{8a^{12}x^{18}}{125y^9}$.

Ex. 15. Extract the cube root—

2. Of $1+6x+12x^2+8x^3$; $\quad 27x^3-54x^2+36x-8$.

3. Of $8a^6-36a^4x^2+54a^2x^4-27x^6$.

4. Of $\frac{x^6}{y^3}-6x^4+12x^2y^3-8y^6$; $\quad \frac{x^3}{8}+\frac{1}{2}+\frac{2}{3x^3}+\frac{8}{27x^6}$.

5. Of $\frac{27x^3}{y^3}-\frac{135x^2}{y^2}+\frac{171x}{y}+55-\frac{114y}{x}-\frac{60y^2}{x^2}-\frac{8y^3}{x^3}$.

6. Of $x^3+\frac{8}{x^3}-12x^2-\frac{48}{x^2}+54x+\frac{108}{x}-112$.

7. Of $(a+1)^{6n}x^3-6ca^p(a+1)^{4n}x^2+12c^2a^{2p}(a+1)^{2n}x-8c^3a^{3p}$.

8. Of $1-x$; $\quad x^2-3a^2$; each to 4 terms.

Ex. 16. Find the value—

1. Of $\left(\frac{x^4}{y^4}-\frac{8x^3}{y^2}+20x^2-8xy^2-26y^4+\frac{8y^6}{x}+\frac{20y^8}{x^2}+\frac{8y^{10}}{x^3}+\frac{y^{12}}{x^4}\right)^{\frac{1}{4}}$.

2. Of $(x^{10}-\frac{5}{2}x^8y+\frac{5}{2}x^6y^2-\frac{5}{4}x^4y^3+\frac{5}{16}x^2y^4-\frac{1}{32}y^5)^{\frac{1}{5}}$.

3. Of $\left(a^6-2a^5b+\frac{5a^4b^2}{3}-\frac{20a^3b^3}{27}+\frac{5a^2b^4}{27}-\frac{2ab^5}{81}+\frac{b^6}{729}\right)^{\frac{1}{6}}$.

SURDS.

Ex. 17. Reduce to entire surds—

1. $2\times 3^{\frac{1}{3}}$; $\quad 25(2\frac{1}{2})^{-\frac{1}{2}}$; $\quad 3\frac{1}{4}(1\frac{1}{12})^{-\frac{3}{2}}$; $\quad \frac{7}{5}(\frac{20}{21})^{\frac{2}{3}}$.

2. $5a^2x^{\frac{1}{2}}$; $\quad -2x\left(\frac{y^2}{x^3}\right)^{\frac{1}{4}}$; $\quad -3a\left(\frac{4bc^2}{27a^3}\right)^{\frac{1}{3}}$; $\quad (x-a)\left(\frac{x+a}{x-a}\right)^{\frac{1}{2}}$.

Ex. 18. Reduce to their most simple forms—

1. $125^{\frac{1}{2}}$; $\quad 162^{\frac{1}{3}}$; $\quad 48^{\frac{1}{4}}$; $\quad 160^{\frac{1}{5}}$; $\quad 3\times 432^{\frac{1}{2}}$.

2. $16^{\frac{2}{3}}$; $\quad (10\frac{1}{8})^{-\frac{3}{4}}$; $\quad 5(2\frac{1}{40})^{\frac{1}{3}}$; $\quad \frac{7}{4}(16\frac{1}{3})^{-\frac{1}{4}}$.

3. $(36a^4x^2y^3)^{\frac{1}{2}}$; $\quad (80a^5b^3c^2)^{\frac{1}{3}}$; $\quad \left(\frac{16x^5y^7}{3}\right)^{\frac{1}{4}}$.

4. $\left(\frac{3xy^3}{98z^4}\right)^{\frac{1}{2}}$; $\quad \frac{2y}{3x}\left(\frac{15x^4z}{4y^3}\right)^{\frac{1}{3}}$; $\quad \frac{a^2b}{3c}\left(\frac{a^7b^4}{9c^5}\right)^{-\frac{1}{4}}$.

Ex. 19. Simplify the expressions—

1. $\sqrt{128}-2\sqrt{50}+\sqrt{72}-\sqrt{18}$.

Ex. 19. Simplify the expressions—

2. $8(\frac{3}{4})^{\frac{1}{2}}+\frac{1}{2}\times 12^{\frac{1}{2}}-\frac{4}{3}\times 27^{\frac{1}{2}}-2(\frac{3}{16})^{\frac{1}{2}}$.

3. $4\sqrt{147}-3\sqrt{75}-6\sqrt{\frac{1}{3}}$; $\quad 40^{\frac{1}{3}}+135^{\frac{1}{3}}$; $\quad 72^{\frac{1}{3}}-3(\frac{1}{3})^{\frac{1}{3}}$.

4. $(16a^2x)^{\frac{1}{2}}+(4a^2x)^{\frac{1}{2}}$; $\quad a(3a^2b)^{\frac{1}{2}}+(12a^4b)^{\frac{1}{2}}$.

5. $b(8a^6b)^{\frac{1}{3}}+4a(a^3b^4)^{\frac{1}{3}}-(125a^6b^4)^{\frac{1}{3}}$; $\quad \frac{3}{4}(8a^3b)^{\frac{1}{3}}-\frac{2b}{3}\left(\frac{a^6}{27b^5}\right)^{\frac{1}{3}}$.

Ex. 20. Find the product—

1. Of $\frac{1}{3}a^{\frac{2}{3}}\times 6a^{\frac{3}{2}}$; $\quad xy(xy)^{\frac{1}{3}}\times x^{-1}y^{\frac{3}{4}}$; $\quad \left(\frac{ay}{x}\right)^{\frac{1}{2}}\cdot\left(\frac{bx}{y^2}\right)^{\frac{1}{3}}\cdot\left(\frac{y^2}{a^3b^2}\right)^{\frac{1}{6}}$.

2. Of $(1-a^{\frac{4}{3}}x^{-\frac{3}{4}}+a^{\frac{1}{3}}x^{\frac{1}{4}}-a^{-\frac{2}{3}}x^{\frac{5}{4}})\times a^{\frac{3}{4}}x^{-\frac{1}{2}}$.

3. Of $(x+2y^{\frac{1}{2}}+3z^{\frac{1}{3}})\times(x-2y^{\frac{1}{2}}+3z^{\frac{1}{3}})$.

4. Of $(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})\times(a^{\frac{1}{3}}-b^{\frac{1}{3}})$; $\quad (x^{\frac{1}{4}}y+y^{\frac{2}{3}})\times(x^{\frac{1}{4}}-y^{-\frac{1}{3}})$.

5. Of $(a^{\frac{5}{2}}+2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}+8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}+32b^{\frac{5}{3}})\times(a^{\frac{1}{2}}-2b^{\frac{1}{3}})$.

6. Of $\{x^{\frac{4}{3}}-2(xy)^{\frac{2}{3}}+y^{\frac{4}{3}}\}\times(x^{\frac{2}{3}}-y^{\frac{2}{3}})$.

7. Of $\{(xy^2)^{\frac{1}{3}}+(xy^3)^{\frac{1}{4}}+(xy^4)^{\frac{1}{5}}\}\times\{(x^2y)^{\frac{1}{3}}-(x^4y)^{\frac{1}{5}}\}$.

8. Of $(\frac{1}{2}a^{\frac{1}{2}}+\frac{1}{3}a^{\frac{1}{3}}+\frac{1}{4}a^{\frac{1}{4}})\times(a^{\frac{1}{4}}-\frac{3}{5}a^{\frac{1}{5}})$.

9. Of $[(a^{-\frac{1}{2}})^{\frac{1}{3}}+\{(a^{\frac{1}{2}}b)^{\frac{1}{2}}\}^{\frac{1}{3}}]\times[(a^{-\frac{1}{2}})^{\frac{1}{3}}-\{(a^{\frac{1}{2}}b)^{\frac{1}{3}}\}^{\frac{1}{2}}]$.

10. Of $\{pax^{\frac{5}{3}}+(p-1)a^2x^{\frac{2}{3}}+(p-2)a^3x^{-\frac{1}{3}}\}\times\left(\frac{1}{a}x^{\frac{4}{3}}-x^{\frac{1}{3}}\right)$.

11. Of $(a^{\frac{5}{8}}+a^{\frac{1}{2}}x^{-\frac{1}{8}}+a^{\frac{3}{8}}x^{-\frac{1}{4}}+a^{\frac{1}{4}}x^{-\frac{3}{8}}+a^{\frac{1}{8}}x^{-\frac{1}{2}}+x^{-\frac{5}{8}})\times$
$(a^{\frac{3}{8}}-a^{\frac{1}{4}}x^{-\frac{1}{8}}+a^{\frac{1}{8}}x^{-\frac{1}{4}}-x^{-\frac{3}{8}})$.

12. Of $(a+b)^{\frac{1}{m}}\times(a+b)^{\frac{1}{n}}\times(a-b)^{\frac{1}{m}}\times(a-b)^{\frac{1}{n}}$.

13. Of $(4+2\sqrt{2})(1-\sqrt{3})(4-2\sqrt{2})(\sqrt{2}+\sqrt{3})(1+\sqrt{3})(\sqrt{2}-\sqrt{3})$.

14. Of $(x-1+2^{\frac{1}{2}})(x-1-2^{\frac{1}{2}})(x+2+3^{\frac{1}{2}})(x+2-3^{\frac{1}{2}})$.

15. Of $(-a)^{\frac{1}{2}}\times(-b)^{\frac{1}{2}}$; $\quad (-a)^{\frac{1}{4}}\times(-b)^{\frac{1}{4}}$; $\quad (-a)^{\frac{1}{6}}\times(-b)^{\frac{1}{6}}$.

16. Of $\{5+2(-3)^{\frac{1}{2}}\}\times\{2-(-3)^{\frac{1}{2}}\}$.

Ex. 21.

1. Divide $\frac{5}{14}(\frac{2}{3})^{\frac{1}{3}}$ by $\frac{5}{21}(\frac{9}{4})^{\frac{1}{3}}$; $\quad \frac{1}{10}(\frac{5}{9})^{\frac{1}{3}}$ by $\frac{9}{5}(\frac{15}{18})^{\frac{1}{2}}$.

2. Divide $15x^{\frac{4}{3}}$ by $35x^{\frac{1}{4}}$; $\quad (x^3y^{mn})^{\frac{1}{m}}$ by $(x^2y^{mn})^{\frac{1}{n}}$; $\quad a^{\frac{2}{3}}\times a^{\frac{7}{5}}$ by a^3.

Ex. 21.

3. Divide $(\frac{4}{3}a^{\frac{3}{2}})^{\frac{1}{2}}-(\frac{3}{4}a^{\frac{3}{2}})^{\frac{1}{2}}$ by $3^{\frac{3}{2}}a^{-\frac{1}{2}}$.

4. Divide $16x-y^2$ by $2x^{\frac{1}{4}}-y^{\frac{1}{2}}$; $\quad 16x-\frac{1}{16}y^4$ by $2x^{\frac{1}{4}}-\frac{1}{2}y$.

5. Divide a^2-b by $a^{\frac{1}{3}}-b^{\frac{1}{6}}$; $\quad a^3-b^3$ by $a^{\frac{3}{4}}+b^{\frac{3}{4}}$.

6. Divide x^2-16y^2 by $x^{\frac{1}{2}}-2y^{\frac{1}{2}}$.

7. Divide $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{3}}$.

8. Divide $x^3-2(x^2y^3)^{\frac{1}{2}}-x^2(x^3y^2)^{\frac{1}{6}}+2y^{\frac{13}{6}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{3}}$.

9. Divide $x^{\frac{5}{2}}-4x^{\frac{3}{2}}-2x^{\frac{1}{2}}-x^2+6x$ by $x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2$.

10. Divide $a-b^2$ by $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$.

11. Divide $x^{-1}-y^{-1}$ by $x^{-\frac{1}{3}}-y^{-\frac{1}{3}}$; $\quad x^{-3}-64y^2$ by $x^{-\frac{1}{2}}+2y^{\frac{1}{3}}$.

12. Divide $8x^{\frac{3}{2}}+y^{-\frac{3}{2}}-z+6x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{1}{3}}$ by $2x^{\frac{1}{2}}+y^{-\frac{1}{2}}-z^{\frac{1}{3}}$.

13. Divide $(p^{10}q^9)^{\frac{1}{12}}-rp^{\frac{7}{10}}q^{\frac{5}{6}}-\frac{3}{2}pq^{\frac{3}{4}}+\frac{3}{2}pqr(p^{-4}q^{-5})^{\frac{1}{30}}$
by $(pq)^{\frac{1}{2}}-\frac{3}{2}(p^4q^3)^{\frac{1}{6}}$.

14. Divide $x^{\frac{3}{2}}-\frac{1}{2}x^{\frac{1}{2}}$ by $x^2+\frac{1}{2}$, to 4 terms in the quotient.

15. Divide $x^{\frac{3}{4}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{3}}$, to 4 terms in the quotient.

16. Divide $x-a$ by $x^{\frac{1}{n}}-a^{\frac{1}{n}}$.

17. Divide $6x(x-y)y^{\frac{1}{2}}-xy(6x)^{\frac{1}{2}}$ by $2x(3^{\frac{1}{2}})-3(2xy)^{\frac{1}{2}}$.

Ex. 22. Simplify the expressions—

1. $\{ab^2.(ab^3)^{\frac{1}{2}}.(ab^4)^{\frac{1}{3}}.(ab^5)^{\frac{1}{4}}\}^{\frac{1}{5}}$; $\quad (a^{\frac{1}{2}})^{\frac{2}{3}-\frac{1}{6}}+6\{a^3b(a^3bc)^{\frac{1}{3}}\}^{\frac{1}{4}}$.

2. $\dfrac{1}{a-(a^2-x^2)^{\frac{1}{2}}}-\dfrac{1}{a+(a^2-x^2)^{\frac{1}{2}}}$; $\quad \left\{\left(\dfrac{a^{-2m}}{b^{-2n}}\right)^{\frac{p}{m}}\right\}^{\frac{q}{2n}}$.

3. $\dfrac{1}{4(1+x^{\frac{1}{2}})}+\dfrac{1}{4(1-x^{\frac{1}{2}})}+\dfrac{1}{2(1+x)}$.

4. $\dfrac{x+(x^2-1)^{\frac{1}{2}}}{x-(x^2-1)^{\frac{1}{2}}}-\dfrac{x-(x^2-1)^{\frac{1}{2}}}{x+(x^2-1)^{\frac{1}{2}}}$.

5. $\dfrac{(x^2+1)^{\frac{1}{2}}+(x^2-1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}-(x^2-1)^{\frac{1}{2}}}+\dfrac{(x^2+1)^{\frac{1}{2}}-(x^2-1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}+(x^2-1)^{\frac{1}{2}}}$.

Ex. 22. Simplify the expressions—

6. $\dfrac{\frac{1}{2}(5^{\frac12}+1)x-2}{x^2-\frac12(5^{\frac12}+1)x+1}-\dfrac{\frac12(5^{\frac12}-1)x+2}{x^2+\frac12(5^{\frac12}-1)x+1}$.

7. $\dfrac{x+(xy^2)^{\frac13}-(x^2y)^{\frac13}}{x+y}$; $\qquad \dfrac{ab}{b-c}\pm\left\{\dfrac{a^2b^2}{(b-c)^2}-\dfrac{a^2b}{b-c}\right\}^{\frac12}$.

8. $\dfrac{1}{x-1}+\dfrac{2}{2x+1-\sqrt{-3}}+\dfrac{2}{2x+1+\sqrt{-3}}$.

9. $a\left(\dfrac{a^3b}{3a^2-6ac+3c^2}\right)^{\frac12}+\dfrac{bc(ab)^{\frac12}}{a-c}$.

10. $\left\{\dfrac{x+y}{x}+\dfrac{x+y}{y}+\dfrac{x^{\frac12}+y^{\frac12}}{x^{\frac12}}+\dfrac{x^{\frac12}-y^{\frac12}}{y^{\frac12}}+\dfrac14\right\}^{\frac{1}{\sqrt2}}$.

Ex. 23.

1. Raise $-2x^{\frac12}y^{\frac13}z^{\frac14}$ to the 3rd, 4th and 6th powers.
2. Raise $-3x^2y^{\frac23}z^{\frac13}$ to the 5th and 9th powers.
3. Raise $\dfrac23\left(\dfrac{3x^2}{4y}\right)^{\frac13}$ to the 4th; $\qquad \{a^3b(a^3bc)^{\frac15}\}^{\frac16}$ to the 7th power.
4. Find the cubes of $a^{\frac13}x^{-1}+a^{-\frac13}x$; and $2x(3y)^{\frac12}-5x^{\frac12}y$.
5. Find the cubes of $\frac23x^{\frac56}y^{-\frac16}-\frac32x^{-\frac16}y^{\frac56}$; and $a^{\frac13}+b^{\frac13}-c^{\frac13}$.
6. Find the cube of $(x+\sqrt{-y^2})^{\frac13}-(x-\sqrt{-y^2})^{\frac13}$.
7. Find the 4th powers of $a^{\frac34}-b^{\frac32}$; and $2x^{\frac32}-3y^{-\frac34}$.

Ex. 24.

1. Find the square roots of $3(5)^{\frac13}$; and $\dfrac{7a^2(a)^{\frac13}}{9(343b^2)^{\frac16}}$.
2. Find the cube roots of $(27a^3x)^{\frac12}$; and $(27a^3x)^{\frac13}$.
3. Find the mth root of $-2^{\frac1m}a^mb^{2m}c^2$.

Ex. 25. Extract the square roots—

1. Of $a^2x^{-2}+2ax^{-1}+3+2a^{-1}x+a^{-2}x^2$; and $-2+a^{2\sqrt2}+a^{-2\sqrt2}$.
2. Of $5x^3-4x(5cx)^{\frac12}+4c$; and $1+\frac{41}{16}a-\dfrac{3+3a}{2}a^{\frac12}+a^2$.

Ex. 25. Extract the square roots—

3. Of $\frac{9}{4}a^3-5a^{\frac{5}{2}}b^{\frac{1}{2}}+\frac{179}{45}a^2b-\frac{4}{3}a^{\frac{3}{2}}b^{\frac{3}{2}}+\frac{4}{25}ab^2$.

4. Of $1+x-\frac{3}{2}x^{\frac{1}{3}}(1+x^{\frac{1}{2}})+x^{\frac{1}{2}}(2+\frac{9}{16}x^{\frac{1}{6}})$.

5. Of $xy-2x(xy-x^2)^{\frac{1}{2}}$; and $ax-2a(ax-a^2)^{\frac{1}{2}}$.

6. Of $a^2+2x(a^2-x^2)^{\frac{1}{2}}$; and $2a+2(a^2-b^2)^{\frac{1}{2}}$.

7. Of $4x^3+xy^2-y^3-4xy(x^2-xy)^{\frac{1}{2}}$; and $1+(1-m^2)^{\frac{1}{2}}$.

8. Of $2+2(1-x)(1+2x-x^2)^{\frac{1}{2}}$; and $x+y+z+2(xz+yz)^{\frac{1}{2}}$.

9. Of $x^m+\frac{1}{4}(b^px^{2n})^{\frac{2}{np}}-b^{\frac{1}{n}}x^{\frac{mp+4}{2p}}$; and $a^{-\frac{2}{xy}}+(a^{\frac{1}{y}}b)^{\frac{1}{r}}-2b^{\frac{1}{2r}}a^{\frac{x-2r}{2xyr}}$.

Ex. 26. Extract the square roots—

1. Of $7+2\sqrt{10}$; $7+4\sqrt{3}$; $2+\sqrt{3}$; $16+6\sqrt{7}$.

2. Of $28-10\sqrt{3}$; $8-2\sqrt{15}$; $41-12\sqrt{5}$; $37-20\sqrt{3}$.

3. Of $4-\sqrt{7}$; $3\sqrt{6}+2\sqrt{12}$; $4\frac{1}{3}-\frac{4}{3}\sqrt{3}$; $3\sqrt{3}+2\sqrt{6}$.

4. Of $\sqrt{18}-\sqrt{16}$; $8\sqrt{3}-6\sqrt{5}$; $\frac{1}{2}\sqrt{18}+2$; $\frac{8}{7}\sqrt{21}-2\sqrt{3}$

Ex. 27. Extract the square roots—

1. Of $4mn+2(m^2-n^2)\sqrt{-1}$; and $2n\sqrt{-1}$.

2. Of $1-4\sqrt{-3}$; $2\sqrt{-3}-2$; $4\sqrt{-5}-1$; $-3+2\sqrt{2}$.

3. Of $21-\sqrt{-400}$; $-3-4\sqrt{-1}$; $\cdot 03\pm\cdot 04\sqrt{-1}$.

4. Of $2\sqrt{-1}$; $-18\sqrt{-1}$.

Ex. 28.

1. Extract the square root of $6+\sqrt{8}-\sqrt{12}-\sqrt{24}$.

2. Extract the square root of $9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15}$.

3. Extract the square root of $12-8\sqrt{2}+6\sqrt{3}-4\sqrt{6}$.

4. Find the value of $(4+3\sqrt{-20})^{\frac{1}{2}}+(4-3\sqrt{-20})^{\frac{1}{2}}$.

Ex. 29. Extract the fourth roots—

1. Of $16a^6-96a^{\frac{9}{2}}x^{\frac{3}{4}}+216a^3x^{\frac{3}{2}}-216a^{\frac{3}{2}}x^{\frac{9}{4}}+81x^3$.

2. Of $x^4y^{-\frac{4}{3}}-4x^{\frac{5}{2}}y^{-\frac{1}{3}}+6xy^{\frac{2}{3}}-4x^{-\frac{1}{2}}y^{\frac{5}{3}}+x^{-2}y^{\frac{8}{3}}$.

Ex. 29. Extract the fourth roots—

3. Of $\frac{1}{16}x^3-\frac{5}{2}x^{\frac{9}{4}}y^{\frac{4}{5}}+\frac{75}{2}x^{\frac{3}{2}}y^{\frac{8}{5}}-250x^{\frac{3}{4}}y^{\frac{12}{5}}+625y^{\frac{16}{5}}$.

4. Of $97+28\sqrt{12}$; $49+20\sqrt{6}$; $14+8\sqrt{3}$; $\frac{17}{3}-4\sqrt{2}$.

5. Of $-79-8\sqrt{-5}$; $-16a^4$; $-64a^8$; -1.

Ex. 30. Find the cube roots—

1. Of $\frac{1}{8}a^3-\frac{3}{2}a^2b^{\frac{1}{2}}+6ab-8b^{\frac{3}{2}}$.

2. Of $a^{-\frac{3}{2}}x^{\frac{3}{2}}-3a^{-1}x+6a^{-\frac{1}{2}}x^{\frac{1}{2}}-7+6a^{\frac{1}{2}}x^{-\frac{1}{2}}-3ax^{-1}+a^{\frac{3}{2}}x^{-\frac{3}{2}}$.

3. Of $54a-32x-36(2ax)^{\frac{1}{3}}(3a^{\frac{1}{3}}-2^{\frac{4}{3}}x^{\frac{1}{3}})$.

4. Of $\{x+(a^2x)^{\frac{1}{3}}\}^2+\{a+(ax^2)^{\frac{1}{3}}\}^2$.

5. Of $38+17\sqrt{5}$; $10\sqrt{7}+22$; $7-5\sqrt{2}$.

6. Of $10-6\sqrt{3}$; $24\sqrt{21}-64$; $11\sqrt{2}+9\sqrt{3}$.

7. Of $-11-2\sqrt{-1}$; $-4-10\sqrt{-2}$; -1.

8. Extract the 5th roots of $41+29\sqrt{2}$; $29\sqrt{2}-41$.

Ex. 31. Reduce to equivalent fractions with rational denominators—

1. $\frac{\sqrt[3]{5\cdot12}+\sqrt[3]{\cdot03375}}{\sqrt[3]{80}-\sqrt[3]{\cdot01}}$; $\frac{1+\sqrt{3}}{2\sqrt{2}-3\sqrt{3}}$; $\frac{1}{2\sqrt{2}-\sqrt{3}}$.

2. $\frac{3}{\sqrt{5}-\sqrt{2}}$; $\frac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$; $\frac{31}{4\sqrt{5}+3\sqrt{2}}$.

3. $\frac{1+\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}$; $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

4. $\frac{1}{\sqrt{3}+\sqrt[3]{5}}$; $\frac{2-\sqrt[3]{3}}{2+\sqrt[3]{3}}$; $\frac{\sqrt[6]{9}-\sqrt{2}}{\sqrt[6]{9}+\sqrt{2}}$; $\frac{1}{a^3-b^{\frac{1}{3}}}$.

Ex. 32. Find the value—

1. Of $\frac{1-ax}{1+ax}\left(\frac{1+bx}{1-bx}\right)^{\frac{1}{2}}$, when $x=\frac{1}{a}\left(\frac{2a}{b}-1\right)^{\frac{1}{2}}$.

2. Of $\frac{2a(1+x^2)^{\frac{1}{2}}}{x+(1+x^2)^{\frac{1}{2}}}$, when $x=\frac{1}{2}\left\{\left(\frac{a}{b}\right)^{\frac{1}{2}}-\left(\frac{b}{a}\right)^{\frac{1}{2}}\right\}$.

Ex. 32. Find the value—

3. Of $\dfrac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}$, when $x=\dfrac{2ab}{b^2+1}$.

4. Of $pq-(1-p^2)^{\frac{1}{2}}(1-q^2)^{\frac{1}{2}}$, when $2p=x+\dfrac{1}{x}$, $2q=y+\dfrac{1}{y}$.

Ex. 33.

1. Which is greater $(\frac{1}{2})^{\frac{1}{2}}$ or $(\frac{2}{3})^{\frac{2}{3}}$; $\sqrt[5]{2}$ or $\sqrt[8]{3}$; $\sqrt[3]{9}$ or $\sqrt[4]{18}$?
2. If $m>3$; prove that $m^{\frac{1}{2}}>(m+1)^{\frac{1}{3}}$.
3. Express $\dfrac{a+b\sqrt{-1}}{c+d\sqrt{-1}}$ in the form $A+B\sqrt{-1}$.
4. Resolve $a^2+b^2+c^2+d^2-2(ad+bc)$ into 2 simple factors.

EQUATIONS.

I. Simple Equations.

Ex. 34. Solve the equations—

1. $3x+5=9x-7$.
2. $x=15x-42$.
3. $5(x+1)-2=3(x+5)$.
4. $3(x-2)+4=4(3-x)$.
5. $13x-21(x-3)=10+21(x-3)$.
6. $5-3(4-x)+4(3-2x)=0$.
7. $3(x-3)-2(x-2)+x-1=x+3+2(x+2)+3(x+1)$.
8. $5(5x-6)-4(4x-5)+3(3x-2)-2x-16=0$.
9. $6x+2x-a=3x+2c$.
10. $\dfrac{x}{a}+\dfrac{x}{b}=c$.
11. $3x+\frac{5}{4}x=34$.
12. $\frac{1}{2}x+\frac{1}{3}x=x-7$.
13. $\frac{1}{2}x-\frac{1}{3}x=\frac{1}{4}x-1$.
14. $\dfrac{x}{2}-\dfrac{x}{3}+\dfrac{x}{4}=2-\dfrac{x}{6}+\dfrac{5x}{12}$.
15. $\dfrac{3x}{4}+\dfrac{7x}{15}+\dfrac{11x}{6}-366=0$.

Ex. 34. Solve the equations—

16. $\frac{x}{2}-\frac{5x+4}{3}=\frac{4x-9}{3}$.

17. $\frac{x+1}{2}+\frac{x+2}{3}=\frac{5-x}{4}+14$.

18. $x-\frac{x-2}{3}=\frac{x+23}{4}-\frac{10+x}{5}$.

19. $\frac{2x-5}{3}+x=\frac{3x-2}{5}+3$.

20. $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36$.

21. $\frac{ax}{b}+\frac{cx}{f}+g=qx+\frac{1}{f}(fh-cx)$.

22. $\frac{2x}{a-2b}=3+\frac{x}{2a-b}$.

23. $\frac{1}{3}(x-a)-\frac{1}{5}(2x-3b)-\frac{1}{2}(a-x)=10a+11b$.

24. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2$.

25. $\frac{a-b}{x-c}=\frac{a+b}{x+2c}$.

26. $\frac{x}{8}-\frac{x-1}{2\frac{1}{2}}=\frac{3x-4}{15}+\frac{x}{12}$.

27. $\frac{x}{6}-8\frac{1}{3}=2\left(\frac{3x}{5}-1\right)-\frac{x+8}{3}+1\frac{2}{3}$.

28. $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$.

29. $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{3}$.

30. $\frac{11x-13}{25}+\frac{19x\quad 3}{7}-\frac{5x-25\frac{1}{3}}{4}=28\frac{1}{7}-\frac{17x+4}{21}$.

31. $1-\frac{x}{2}\left(1-\frac{3}{4x}\right)=\frac{2}{3}\left(3-\frac{5x}{2}\right)+5\frac{3}{8}$.

32. $\frac{3x-\frac{2}{3}(1+}{4}+\frac{1-\frac{1}{5}x}{5\frac{1}{2}}=\frac{2\frac{2}{5}+\frac{1}{25}(x-1)}{2\frac{1}{5}}$.

33. $\frac{1}{2}\left(x-\frac{51}{26}\right)-\frac{2}{13}(1-3x)=x-\frac{1}{39}\left(5x-\frac{1-\cdot 3x}{\cdot 4}\right)$

Ex. 34. Solve the equations—

34. $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}=\dfrac{7}{3}$.

35. $\dfrac{2(3-4x)}{3-x}+\dfrac{3}{1-x}=8$.

36. $\dfrac{x-7}{x+7}=\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}$.

37. $\dfrac{(2x+3)x}{2x+1}+\dfrac{1}{3x}=x+1$.

38. $\dfrac{7x+1}{x-1}=\dfrac{35}{9}\left(\dfrac{x+4}{x+2}\right)+\dfrac{28}{9}$.

39. $\dfrac{66x+1}{1\cdot5x+1}+\dfrac{4x+5}{\cdot5x-1}=52$.

40. $\dfrac{3x^2-2x+1}{5}=\dfrac{(7x-2)(3x-6)}{35}+\dfrac{9}{10}$.

41. $\dfrac{6-5x}{15}-\dfrac{7-2x^2}{14(x-1)}=\dfrac{3x+1}{21}-\dfrac{2x-2\frac{1}{5}}{6}+\dfrac{1}{105}$.

42. $\dfrac{x}{2}-\dfrac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)}=\dfrac{3}{2}\left(\dfrac{x^2+2}{3x-2}\right)$.

43. $\dfrac{3+2x}{1+2x}-\dfrac{5+2x}{7+2x}=1-\dfrac{4x^2-2}{7+16x+4x^2}$.

Ex. 35. Solve the equations—

1. $(12+x)^{\frac{1}{2}}=2+x^{\frac{1}{2}}$.
2. $(5x+10)^{\frac{1}{2}}=(5x)^{\frac{1}{2}}+2$.
3. $(3x^{\frac{1}{2}}+5)^2=108+9x$.
4. $\frac{1}{3}(17x-6)^{\frac{1}{3}}+\frac{3}{8}=1\frac{1}{24}$.
5. $\dfrac{3x-1}{(3x)^{\frac{1}{2}}+1}=1+\dfrac{(3x)^{\frac{1}{2}}-1}{2}$.
6. $\dfrac{ax-b^2}{(ax)^{\frac{1}{2}}+b}-\dfrac{(ax)^{\frac{1}{2}}-b}{c}=c$.
7. $x^{\frac{1}{2}}-(a+x)^{\frac{1}{2}}=\left(\dfrac{a}{x}\right)^{\frac{1}{2}}$.
8. $\dfrac{1}{a}(a+x)^{\frac{1}{2}}+\dfrac{1}{x}(a+x)^{\frac{1}{2}}=\dfrac{1}{b}x^{\frac{1}{2}}$.

Ex. 35. Solve the equations—

9. $a+x\pm(a^2+x^2)^{\frac{1}{2}}=b.$

10. $x^{\frac{1}{2}}+\{x-(1-x)^{\frac{1}{2}}\}^{\frac{1}{2}}=1.$

11. $x^{\frac{1}{2}}+\{a-(ax+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}=a^{\frac{1}{2}}.$

12. $(a-x)^{\frac{1}{2}}+2(a+x)^{\frac{1}{2}}=\{a-x+(ax+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}.$

13. $x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}=\dfrac{na}{(a+x)^{\frac{1}{2}}}.$

14. $x^{\frac{1}{2}}+2(2a-x)^{\frac{1}{2}}=\{x+(2a^2-3ax+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}.$

15. $(1-x)^{\frac{1}{2}}+\{1-x+(1+x)^{\frac{1}{2}}\}^{\frac{1}{2}}=(1+x)^{\frac{1}{2}}.$

16. $(x+x^{\frac{1}{2}})^{\frac{1}{2}}-(x-x^{\frac{1}{2}})^{\frac{1}{2}}=a\left(\dfrac{x}{x+x^{\frac{1}{2}}}\right)^{\frac{1}{2}}.$

17. $(x^{\frac{1}{2}}+3)^{\frac{1}{2}}-(x^{\frac{1}{2}}-3)^{\frac{1}{2}}=(2x^{\frac{1}{2}})^{\frac{1}{2}}.$

18. $(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=2^{\frac{1}{3}}.$

19. $(a+x)^{\frac{1}{m}}=(x^2+8ax+b^2)^{\frac{1}{2m}}.$

20. $(1-a)^{\frac{1}{2}}\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}}+(1+a)^{\frac{1}{2}}\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{4}}=2(1-a^2)^{\frac{1}{4}}.$

21. $ax+1=\dfrac{2ax(x+a^2)^{\frac{1}{2}}}{a+(x+a^2)^{\frac{1}{2}}}.$

22. $\dfrac{a^{\frac{1}{2}}-\{a-(a^2-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}}{a^{\frac{1}{2}}+\{a-(a^2-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}}=b.$

23. $\dfrac{(36x+1)^{\frac{1}{2}}+(36x)^{\frac{1}{2}}}{(36x+1)^{\frac{1}{2}}-(36x)^{\frac{1}{2}}}=9.$

24. $\dfrac{1+x-(2x+x^2)^{\frac{1}{2}}}{1+x+(2x+x^2)^{\frac{1}{2}}}=a\dfrac{(2+x)^{\frac{1}{2}}+x^{\frac{1}{2}}}{(2+x)^{\frac{1}{2}}-x^{\frac{1}{2}}}.$

II. Simultaneous Equations of the First Degree.

Ex. 36. Find the values of x and y in the equations—

1. $\left.\begin{aligned}2x&=11+9y,\\3x-12y&=15.\end{aligned}\right\}$

2. $\left.\begin{aligned}3x-7y&=7,\\11x+5y&=87.\end{aligned}\right\}$

Ex. 36. Find the values of x and y in the equations—

3. $\left.\begin{array}{l} 9x-4y-8=0, \\ 13x+7y-101=0. \end{array}\right\}$

4. $\left.\begin{array}{l} y(3+x)=x(7+y), \\ 4x+9\ \ =5y-14. \end{array}\right\}$

5. $\left.\begin{array}{l} ax=by, \\ x+y=c. \end{array}\right\}$

6. $\left.\begin{array}{l} x+ay=b, \\ ax-by=c. \end{array}\right\}$

7. $\left.\begin{array}{l} 3ax-2by=c, \\ a^2x+b^2y=5bc. \end{array}\right\}$

8. $\left.\begin{array}{l} (a-b)x+(a+b)y=c, \\ (a^2-b^2)(x+y)=n. \end{array}\right\}$

9. $\left.\begin{array}{l} \dfrac{x}{3}+\dfrac{y}{5}=8, \\ \dfrac{x}{9}-\dfrac{y}{10}=1. \end{array}\right\}$

10. $\left.\begin{array}{l} \dfrac{x}{9}+\dfrac{y}{8}=43, \\ \dfrac{x}{8}+\dfrac{y}{9}=42. \end{array}\right\}$

11. $\left.\begin{array}{l} x-\frac{1}{7}(y-2)=5, \\ 4y-\frac{1}{3}(x+10)=3. \end{array}\right\}$

12. $\left.\begin{array}{l} \dfrac{4x+5y}{40}=x-y, \\ \dfrac{2x-y}{3}+2y=\frac{1}{2}. \end{array}\right\}$

13. $\left.\begin{array}{l} \frac{1}{3}(x+y)+\frac{1}{4}(x-y)=59, \\ 5x-33y=0. \end{array}\right\}$

14. $\left.\begin{array}{l} 2x-\dfrac{y+3}{4}=7+\dfrac{3y-2x}{5}, \\ 4y+\dfrac{x-2}{3}=26\frac{1}{2}-\dfrac{2y+1}{2}. \end{array}\right\}$

15. $\left.\begin{array}{l} \dfrac{3x+4y+3}{10}-\dfrac{2x+7-y}{15}=5+\dfrac{y-8}{5}, \\ \dfrac{9y+5x-8}{12}-\dfrac{x+y}{4}=\dfrac{7x+6}{11}. \end{array}\right\}$

16. $\left.\begin{array}{l} 2x+{\cdot}4y=1{\cdot}2, \\ 3{\cdot}4x-{\cdot}02y={\cdot}01. \end{array}\right\}$

17. $\left.\begin{array}{l} \dfrac{1}{3x}+\dfrac{1}{5y}=\dfrac{2}{9}, \\ \dfrac{1}{5x}+\dfrac{1}{3y}=\dfrac{1}{4}. \end{array}\right\}$

18. $\left.\begin{array}{l} \dfrac{m}{x}+\dfrac{n}{y}=a, \\ \dfrac{n}{x}+\dfrac{m}{y}=b. \end{array}\right\}$

19. $\left.\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b}=1-\dfrac{x}{c}, \\ \dfrac{y}{a}+\dfrac{x}{b}=1+\dfrac{y}{c}. \end{array}\right\}$

20. $\left.\begin{array}{l} 6x-\dfrac{2y-x}{23-x}=20-\dfrac{59-12x}{2}, \\ 3y+\dfrac{y-3}{x-18}=30-\dfrac{73-9y}{3}. \end{array}\right\}$

Ex. 36. Find the values of x and y in the equations—

21. $$\left.\begin{aligned}&\frac{4x-2y+3}{3}-\frac{18-x+5y}{7}=\frac{x}{4}-\frac{y}{5}-\frac{1}{7}-7\tfrac{7}{10},\\&(2x-y+15)\left(\frac{y}{4}-\frac{x}{3}+\frac{1}{12}\right)=(y-2x+15)\left(\frac{x}{3}-\frac{y}{4}+\frac{3}{4}\right).\end{aligned}\right\}$$

22. $$\left.\begin{aligned}&\frac{4x-8y+5}{2}=\frac{10x^2-12y^2-14xy+2x}{5x+3y+3}+2,\\&2(6+x)^{\frac{1}{2}}=3(6-y)^{\frac{1}{2}}.\end{aligned}\right\}$$

23. $$\left.\begin{aligned}&(a^2-b^2)(5x+3y)=(4a-b)2ab,\\&a^2y-\frac{ab^2c}{a+b}+(a+b+c)bx=b^2y+(a+2b)ab.\end{aligned}\right\}$$

24. $$\left.\begin{aligned}&y^{\frac{1}{2}}-(20-x)^{\frac{1}{2}}=(y-x)^{\frac{1}{2}},\\&3(20-x)^{\frac{1}{2}}=2(y-x)^{\frac{1}{2}}.\end{aligned}\right\}$$

Ex. 37. Find the values of x, y, z, &c. in the equations—

1. $$\left.\begin{aligned}2x+4y+5z&=49,\\3x+5y+6z&=64,\\4x+3y+4z&=55.\end{aligned}\right\}$$

2. $$\left.\begin{aligned}x+2y+3z&=17,\\2x-3y+z&=0,\\3x+y-5z&=-15.\end{aligned}\right\}$$

3. $$\left.\begin{aligned}3x-7y+4z&=1,\\-5x+9y-z&=22,\\x-2y+z&=0.\end{aligned}\right\}$$

4. $$\left.\begin{aligned}5x+3y&=65,\\2y-z&=11,\\3x+4z&=57.\end{aligned}\right\}$$

5. $$\left.\begin{aligned}4(y-x)&=5z-22,\\3z+4x&=6y+2,\\z-3y&=14-10x.\end{aligned}\right\}$$

6. $$\left.\begin{aligned}2(x-y)&=3z-2,\\x-3z&=3y-1,\\2x+3z&=4(1-y).\end{aligned}\right\}$$

7. $$\left.\begin{aligned}&\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12,\\&\frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8,\\&\frac{x}{2}+\frac{z}{3}=10.\end{aligned}\right\}$$

8. $$\left.\begin{aligned}&y+\frac{z}{3}=\frac{x}{5}+5,\\&\frac{x-1}{4}-\frac{y-2}{5}=\frac{z+3}{10},\\&x-\frac{2y-5}{3}=1\tfrac{1}{4}-\frac{z}{12}.\end{aligned}\right\}$$

9. $$\left.\begin{aligned}&\frac{2x+z-4}{12}+\frac{3y-6z+1}{13}=\frac{x-2}{4},\\&\frac{3x-2y+5}{5}-\frac{4x-5y+7z}{7}=\frac{2}{7}+\frac{3y-9z+6}{6},\\&\frac{x}{9}-y+3z=2.\end{aligned}\right\}$$

Ex. 37. Find the values of x, y, z, &c. in the equations—

10. $\left.\begin{aligned}&x-y+z=0,\\&(a+b)x-(a+c)y+(b+c)z=0,\\&abx-acy+bcz=1.\end{aligned}\right\}$

11. $\left.\begin{aligned}&\frac{1}{x}+\frac{1}{y}=a,\\&\frac{1}{x}+\frac{1}{z}=b,\\&\frac{1}{y}+\frac{1}{z}=c.\end{aligned}\right\}$

12. $\left.\begin{aligned}&\frac{xy}{x+y}=1,\\&\frac{xz}{x+z}=2,\\&\frac{yz}{y+z}=3.\end{aligned}\right\}$

13. $\left.\begin{aligned}xy&=3(x+y),\\xz&=8(x+z),\\7yz&=9(y+z).\end{aligned}\right\}$

14. $\left.\begin{aligned}&\frac{10}{x}-\frac{4}{5y}+\frac{1}{z}=21\cdot6,\\&\frac{1}{3x}+\frac{5}{9y}+\frac{2}{z}=10\cdot\dot{3},\\&\frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=16\cdot1.\end{aligned}\right\}$

15. $\left.\begin{aligned}&\frac{2}{x}+\frac{3}{y}-\frac{4}{z}=\frac{1}{12},\\&\frac{3}{x}-\frac{4}{y}+\frac{5}{z}=\frac{19}{24},\\&-\frac{4}{x}+\frac{5}{y}+\frac{6}{z}=\frac{1}{2}.\end{aligned}\right\}$

16. $\left.\begin{aligned}&5x-11y^{\frac{1}{2}}+13z^{\frac{1}{3}}=22,\\&4x+6y^{\frac{1}{2}}+5z^{\frac{1}{3}}=31,\\&x-y^{\frac{1}{2}}+z^{\frac{1}{3}}=2.\end{aligned}\right\}$

17. $\left.\begin{aligned}9x-2z+u&=41,\\7y-5z-t&=12,\\4y-3x+2u&=5,\\3y-4u+3t&=7,\\7z-5u&=11.\end{aligned}\right\}$

III. Problems in Equations of the First Degree.

Ex. 38.

1. What number is that, to which if 16 be added, 4 times the sum will be equal to 10 times the number increased by one?

2. What number is that which exceeds its seventh part by 12?

3. Find that number to which if a third part of it be added, the sum will equal 4 times the number diminished by 8.

4. Find a number such that if increased by 16, it will become 7 times as great as the third part of the original number.

5. The sum of two numbers is 14; and half their difference is 2: find the numbers.

6. A and B begin to play with equal sums; A won £5 and then 3 times A's money was equal to 11 times B's; what had each at first?

7. A is twice as old as B; 22 years ago he was 3 times as old. What is A's age?

8. A messenger starts on an errand at the rate of 4 miles an

Ex. 38.

hour; another is sent $1\frac{1}{2}$ hours after to overtake him; the latter walks at the rate of $4\frac{1}{4}$ miles an hour; when and where will he overtake him?

9. A and B engaged in play; when A had lost £20 he had only one-third of the money which B had; but by continuing to play he not only won back his £20 but also £50 more with it, and then found he possessed half as much again as B: with what sum did they respectively begin?

10. A garrison of 500 men was victualled for 48 days; after 15 days it was reinforced, and then the provisions were exhausted in 11 days: required the number of men in the reinforcement.

11. If A does a piece of work in 10 days, which A and B can do together in 7 days; how long would B take to do it alone?

12. A person performs two-sevenths of a piece of work in 13 days; he then, with the aid of another person, finishes it in 6 days; in what time could each do it separately?

13. Find the time in which A, B and C can together perform a piece of work, which requires 7, 6 and 9 days respectively when done singly.

14. A cistern is filled in 24 minutes by 3 pipes, one of which conveys 8 gallons more, and another 7 gallons less than the third, every 3 minutes. The cistern holds 1050 gallons. How much flows through each pipe in a minute?

15. A person buys 4 houses: for the 2nd he gives half as much again as for the 1st; for the 3rd, half as much again as for the 2nd; and for the 4th as much as for the 1st and 3rd together: he pays £8000 for them all. What is the cost of each?

16. Find the fraction, which, if 1 be added to its numerator, becomes $\frac{1}{3}$; but if 1 be added to its denominator, becomes $\frac{1}{4}$.

17. Find 2 numbers, such that if the first be added to 4 times the second, the sum is 29; and if the second be added to 6 times the first, the sum is 36.

18. Find that number of 2 figures, to which, if the number formed by changing the places of the digits be added, the sum is 121; and if the same 2 numbers be subtracted, the remainder is 9.

19. A man and his wife could drink a barrel of beer in 15 days; after drinking together 6 days, the woman alone drank the remainder in 30 days: in what time could either alone drink it all?

20. To complete a certain work, A requires m times as many days as B and C together; B requires n times as many as A and C

Ex. 38.

together; and C requires p times as many as A and B together: compare the times in which each would do it; and prove that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

21. From a certain sum I took away a third part, and put in its stead £50; next from the sum thus augmented I took away one-fourth, and put in its stead £70; I then counted the money and found £120: what was the original sum?

22. A bill of 25 guineas was paid with crowns and half-guineas; and twice the number of half-guineas exceeded 3 times that of the crowns by 17: how many were there of each?

23. A and B start to run a race to a certain post and back again. A returning meets B at 90 yards from the post and arrives at the starting-place 3 minutes before him. If he had returned immediately to meet B, he would have met him at one-sixth of the distance between the post and the starting-place. Find the length of the course and the duration of the race.

24. A farmer sells to one person 9 horses and 7 cows for £300 and to another, at the same prices, 6 horses and 13 cows for the same sum: what was the price of each?

25. A farmer mixes barley at 2*s.* 4*d.* a bushel with rye at 3*s.* a bushel, and wheat at 4*s.* a bushel, so that the whole is 100 bushels and worth 3*s.* 4*d.* a bushel. Had he put twice as much rye, and 10 bushels more of wheat, the whole would have been worth exactly the same per bushel: how much of each kind was there?

IV. Pure Quadratic Equations.

Ex. 39. Solve the equations—

1. $x^2+5=\frac{10}{3}x^2-16.$

2. $\frac{3}{4}x^2-(2x^2-3)=\frac{16x^2+9}{5}.$

3. $8x+\frac{7}{x}=\frac{65}{7}x.$

4. $15x-\frac{11}{x}=\frac{13}{5}x.$

5. $\frac{8}{1-2x}+\frac{8}{1+2x}=25.$

6. $3\left(\frac{x^2-9}{x^2+3}\right)+4\left(\frac{22\frac{1}{2}+x^2}{x^2+9}\right)=7.$

7. $x(x-10)=(6\frac{2}{5}-x)10.$

8. $(5x+\frac{1}{2})^2=756\frac{1}{4}+5x.$

9. $\frac{7x^2+8}{21}-\frac{x^2+4}{8x^2-11}=\frac{x^2}{3}.$

10. $\frac{35-2x}{9}+\frac{5x^2+7}{5x^2-7}=\frac{17-\frac{2}{3}x}{3}.$

Ex. 39. Solve the equations—

11. $\dfrac{a}{b+x}+\dfrac{a}{b-x}=c.$

12. $x(6+x^2)^{\frac{1}{2}}=1+x^2.$

13. $(1+x+x^2)^{\frac{1}{2}}=a-(1-x+x^2)^{\frac{1}{2}}.$

14. $\dfrac{(a+bx^2)^{\frac{1}{2}}+(a-bx^2)^{\frac{1}{2}}}{(a+bx^2)^{\frac{1}{2}}-(a-bx^2)^{\frac{1}{2}}}=c.$

15. $\dfrac{a+x+(a^2-x^2)^{\frac{1}{2}}}{a+x-(a^2-x^2)^{\frac{1}{2}}}=\dfrac{b}{x}.$

16. $\dfrac{ax+1+(a^2x^2-1)^{\frac{1}{2}}}{ax+1-(a^2x^2-1)^{\frac{1}{2}}}=\dfrac{b^2x}{2}.$

17. $\dfrac{1+x^3}{(1+x)^2}+\dfrac{1-x^3}{(1-x)^2}=a.$

18. $\dfrac{1-ax}{1+ax}\left(\dfrac{1+bx}{1-bx}\right)^{\frac{1}{2}}=1.$

19. $\dfrac{a+x}{a^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}}+\dfrac{a-x}{a^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}=a^{\frac{1}{2}}.$

20. $\{a+(a^2-x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}+\{a-(a^2-x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}=n\left\{\dfrac{a+x}{a+(a^2-x^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$

21. $\dfrac{(1+x)^{\frac{1}{2}}-1}{(1-x)^{\frac{1}{2}}+1}+\dfrac{(1-x)^{\frac{1}{2}}+1}{(1+x)^{\frac{1}{2}}-1}=a.$

22. $\{(1+x)^2-ax\}^{\frac{1}{2}}+\{(1-x)^2+ax\}^{\frac{1}{2}}=x.$

V. Adfected Quadratic Equations.

Ex. 40. Solve the equations—

1. $x^2-10x=24.$
2. $x^2+2x=80.$
3. $x^2-18x+32=0.$
4. $x^2+10=13(x+6).$
5. $x^2-x=11342.$
6. $x^2+9x-52=0.$
7. $x^2-111x=3400.$
8. $x^2=5(x+89)+5555.$
9. $x^2-\frac{9}{5}x+\frac{9}{20}=0.$
10. $x^2-\frac{5}{3}x-4=0.$
11. $\dfrac{x^2}{3}=9+\dfrac{x}{2}.$
12. $4x^2-4x=80.$
13. $6x^2+5x-4=0.$
14. $12x^2-x-1740=0.$
15. $3x^2-12x+1=6x-23.$
16. $11x^2-11\frac{1}{4}=9x.$
17. $17x^2+19x=1848.$
18. $3(x-2)^2=18+(8x+1).$
19. $x-\dfrac{x^3-8}{x^2+5}=2.$
20. $x+\dfrac{x^2+3}{x^2-5}=\dfrac{12+5x^3}{5(x^2-5)}.$
21. $\dfrac{21x^3-16}{3x^2-4}-7x=5.$
22. $\dfrac{x+16}{5}+\dfrac{11}{x}=\dfrac{4x-17\frac{1}{3}}{3}.$

Ex. 40. Solve the equations—

23. $\dfrac{4x}{9}+\dfrac{x-5}{x+3}=\dfrac{4x+7}{19}.$

24. $\dfrac{4}{x-3}-\dfrac{3}{x+5}=\dfrac{1}{18}.$

25. $\dfrac{3x}{x+1}+\dfrac{2x-5}{3x-1}=3\tfrac{10}{69}.$

26. $\dfrac{7}{x^2+4x}+\dfrac{21}{3x^2-8x}=\dfrac{22}{x}.$

27. $x^2-(a+b)x+ab=0.$

28. $(a-b)x^2-(a+b)x+2b=0.$

29. $\dfrac{x^2}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-(a^{\frac{1}{2}}-b^{\frac{1}{2}})x=\dfrac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}.$

30. $mqx^2-mnx+pqx-np=0.$

31. $\dfrac{2x(a-x)}{3a-2x}=\dfrac{a}{4}.$

32. $\dfrac{16}{x^{\frac{3}{2}}}+\dfrac{x^{\frac{1}{2}}}{2}=\dfrac{6}{x^{\frac{1}{2}}}.$

33. $\dfrac{1}{a}+\dfrac{1}{a+x}+\dfrac{1}{a+2x}=0.$

34. $3x^2=5x^4-8x^2-306.$

35. $x^6-7x^3=8.$

36. $11-9x^3+x^6=299+3x^3-5x^6.$

37. $x^{-2}-2x^{-1}=8.$

38. $x^{-1}+ax^{-\frac{1}{2}}=2a^2.$

39. $3x^{\frac{1}{2}}-x^{-\frac{1}{2}}+2=0.$

40. $x^{-\frac{1}{3}}+2=\dfrac{x^{-1}+8}{x^{-\frac{2}{3}}+5}.$

41. $x^{\frac{7}{3}}+\dfrac{41x^{\frac{1}{3}}}{x}=\dfrac{97}{x^{\frac{2}{3}}}+x^{\frac{5}{6}}.$

42. $(x-c)(ab)^{\frac{1}{2}}-(a-b)(cx)^{\frac{1}{2}}=0.$

43. $x+5=(x+5)^{\frac{1}{2}}+6.$

44. $x^2=21+(x^2-9)^{\frac{1}{2}}.$

45. $x^2-2x+6(x^2-2x+5)^{\frac{1}{2}}=11.$

46. $x^2-x+5(2x^2-5x+6)^{\frac{1}{2}}=\frac{1}{2}(3x+33).$

47. $9x-3x^2+4(x^2-3x+5)^{\frac{1}{2}}=11.$

48. $x+(x^2-ax+b^2)^{\frac{1}{2}}=\dfrac{x^2}{a}+b.$

49. $\dfrac{(x^2+x+6)^{\frac{1}{2}}}{3}=\dfrac{20-\frac{4}{3}(x^2+x+6)^{\frac{1}{2}}}{(x^2+x+6)^{\frac{1}{2}}}.$

50. $x+4+\left(\dfrac{x+4}{x-4}\right)^{\frac{1}{2}}=\dfrac{12}{x-4}.$

51. $\{(x-2)^2-x\}^2-(x-2)^2=88-(x-2).$

52 $\{x+(2x-1)^{\frac{1}{2}}\}^{\frac{1}{2}}-\{x-(2x-1)^{\frac{1}{2}}\}^{\frac{1}{2}}=\dfrac{3}{5}\left\{\dfrac{10x}{x+(2x-1)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$

Ex. 40. Solve the equations—

53. $\dfrac{x}{a+x}+\dfrac{a}{(a+x)^{\frac{1}{2}}}=\dfrac{b}{x}$.

54. $cx=\{(1+x)^{\frac{1}{2}}-1\}\{(1-x)^{\frac{1}{2}}+1\}$.

55. $\dfrac{x}{8}+\dfrac{1}{2x}=\left(\dfrac{x}{3}+\dfrac{1}{4}\right)^{\frac{1}{2}}-\dfrac{2}{3}$.

56. $(x+2)^2+2(x+2)x^{\frac{1}{2}}-3x^{\frac{1}{2}}=46+2x$.

57. $\dfrac{x^2}{4}=\dfrac{x-12}{x^2-18}$.

58. $\dfrac{x^2}{2x-5}=\dfrac{3}{x^2-17}$.

59. $x-3=\dfrac{3+4x^{\frac{1}{2}}}{x}$.

60. $x^{\frac{1}{2}}-\dfrac{8}{x}=\dfrac{7}{x^{\frac{1}{2}}-2}$.

61. $4x^2+12x(1+x)^{\frac{1}{2}}=27(1+x)$.

62. $2x^2+(x^2+9)^{\frac{1}{2}}=x^4-9$.

63. $(a+x)^{\frac{2}{3}}+4(a-x)^{\frac{2}{3}}-5(a^2-x^2)^{\frac{1}{3}}=0$.

64. $\dfrac{a+x}{(a-x)^{\frac{1}{2}}}+\dfrac{a-x}{(a+x)^{\frac{1}{2}}}=2a^{\frac{1}{2}}$.

65. $(1+x)^{\frac{2}{5}}+(1-x)^{\frac{2}{5}}=(1-x^2)^{\frac{1}{5}}$.

66. $\dfrac{(1+x)^2}{1+x^3}+\dfrac{(1-x)^2}{1-x^3}=a$.

67. $\dfrac{1+x^3}{(1+x)^3}+\dfrac{1-x^3}{(1-x)^3}=c$.

68. $(a+x)^{\frac{1}{4}}+(a-x)^{\frac{1}{4}}=h$.

69. $2x(1-x^4)^{\frac{1}{2}}=a(1+x^4)$.

70. $\dfrac{x+(x^2-a^2)^{\frac{1}{2}}}{x-(x^2-a^2)^{\frac{1}{2}}}=\dfrac{x}{a}$.

71. $x^{\frac{p+q}{2pq}}=\dfrac{1}{2}\,\dfrac{a^2-b^2}{a^2+b^2}(x^{\frac{1}{p}}+x^{\frac{1}{q}})$.

72. $\dfrac{(27a+8x)^{\frac{2}{3}}}{15x^{\frac{13}{15}}}+\dfrac{8x^{\frac{2}{15}}}{3(27a+8x)^{\frac{1}{3}}}=\dfrac{8}{5x^{\frac{1}{5}}}$.

Ex. 41. Solve the equations—

1. $x+7x^{\frac{1}{3}}=22$. [N.B. $x-8+7(x^{\frac{1}{3}}-2)=0$.]

2. $x^3-3x=2$. [N.B. $x(x^2-4)+(x-2)=0$.]

3. $2x^3-x^2=1$. [N.B. $2x^2(x-1)+x^2-1=0$.]

4. $x^3-6x+9=0$. [N.B. $x(x^2-9)+3(x+3)=0$.]

Ex. 41. Solve the equations—

5. $x^3-6x^2+10x-8=0.$ [N.B. $x(x^2-6x+8)+2(x-4)=0.$]

6. $27x^3-135x^2+225x-117=0.$ [N.B. $3x-3$ is a divisor.]

7. $x^2-\frac{27}{4}x+25=7x^{\frac{1}{2}}(5-x).$ [N.B. $(x-5)^2-7x^{\frac{1}{2}}(x-5)+\frac{49}{4}x=9x.$]

8. $x^4-2x^3+x=132.$ [N.B. $x^4-2x^3+x^2-(x^2-x)=132.$]

9. $x^4+x^3-4x^2+x+1=0.$

10. $x^4-x^3+\frac{5}{4}x^2-x+1=0.$

11. $x^4-8x^3+10x^2+24x+5=0.$ [N.B. $(x^2-4x)^2-6(x^2-4x)+5=0.$]

12. $x^4+\frac{13}{3}x^3-39x=81.$ [N.B. $x^4-81+\frac{13}{3}(x^2-9)x=0.$]

13. $\dfrac{1+x^3}{(1+x)^3}=h.$ 14. $\dfrac{1+x^4}{(1+x)^4}=\dfrac{1}{2}.$ 15. $\dfrac{1+x^5}{(1+x)^5}=a.$

VI. Simultaneous Equations of the 2nd, 3rd, &c. Degrees.

Ex. 42. Solve the equations—

1. $\left.\begin{array}{l} x-2y=2, \\ 3xy=36. \end{array}\right\}$ 2. $\left.\begin{array}{l} x^2+y^2=85, \\ xy=42. \end{array}\right\}$

3. $\left.\begin{array}{l} x^2+y^2=41, \\ x-y=1. \end{array}\right\}$ 4. $\left.\begin{array}{l} x^2+y^2=10, \\ x+4y=7. \end{array}\right\}$

5. $\left.\begin{array}{l} x^2-6xy+11y^2=9, \\ x-3y=1. \end{array}\right\}$ 6. $\left.\begin{array}{l} x^2-xy=153, \\ x+y=1. \end{array}\right\}$

7. $\left.\begin{array}{l} x^2-xy-y^2=\frac{1}{15}xy, \\ x-y=2. \end{array}\right\}$ 8. $\left.\begin{array}{l} 2x^2-3xy+y^2=24, \\ 3x^2-5xy+2y^2=33. \end{array}\right\}$

9. $\left.\begin{array}{l} 7x^2-8xy=159, \\ 5x+2y=7. \end{array}\right\}$ 10. $\left.\begin{array}{l} 16x^2+xy=203\frac{7}{8}, \\ 6x-8y=3. \end{array}\right\}$

11. $\left.\begin{array}{l} 5xy+3x-y=492, \\ 2y=3x. \end{array}\right\}$ 12. $\left.\begin{array}{l} \frac{1}{5}(3x+5y)+\frac{1}{3}(4x-3y)=13\frac{8}{15}, \\ 3x^2+2y^2=179. \end{array}\right\}$

13. $\left.\begin{array}{l} \dfrac{x^2}{y^2}=\dfrac{85y-36x}{9y}, \\ x-y=2. \end{array}\right\}$ 14. $\left.\begin{array}{l} 5xy=84-x^2y^2, \\ x-y=6. \end{array}\right\}$

15. $\left.\begin{array}{l} \frac{1}{10}(x^2+y^2)=\frac{1}{3}(x+y), \\ xy=8. \end{array}\right\}$ 16. $\left.\begin{array}{l} 14xy=(2x-\frac{3}{7})(7y+\frac{1}{2}), \\ x^2y^2=(x^2-9)(y^2+1). \end{array}\right\}$

17. $\left.\begin{array}{l} x+y+xy=11, \\ x^2y+xy^2=30. \end{array}\right\}$ 18. $\left.\begin{array}{l} x+3(x+y)^2=310-y, \\ xy=21. \end{array}\right\}$

19. $\left.\begin{array}{l} x^2+xy+y^2=52, \\ xy-x^2=8. \end{array}\right\}$ 20. $\left.\begin{array}{l} x^2+xy=a^2, \\ y^2-xy=b^2. \end{array}\right\}$

Ex. 42. Solve the equations—

21. $\left.\begin{aligned} x^{-1}+y^{-1}&=a^{-1}, \\ x^{-2}+y^{-2}&=b^{-2}. \end{aligned}\right\}$

22. $\left.\begin{aligned} &x^2-(y+1)x+(y-1)y=51, \\ &x(y+1)=39-y. \end{aligned}\right\}$

23. $\left.\begin{aligned} x^{\frac{1}{2}}+y^{\frac{1}{2}}&=6, \\ x+y&=20. \end{aligned}\right\}$

24. $\left.\begin{aligned} &(6x^{\frac{1}{2}}+6y^{\frac{1}{2}})^{\frac{1}{2}}+\tfrac{1}{2}x^{\frac{1}{2}}=9-\tfrac{1}{2}y^{\frac{1}{2}}, \\ &x-y=12. \end{aligned}\right\}$

25. $\left.\begin{aligned} x^3-y^3&=63, \\ x-y&=3. \end{aligned}\right\}$

26. $\left.\begin{aligned} x^{\frac{1}{3}}+y^{\frac{1}{3}}&=5, \\ x+y&=35. \end{aligned}\right\}$

27. $\left.\begin{aligned} x^{\frac{1}{2}}+y^{\frac{1}{2}}&=4, \\ x^{\frac{3}{2}}+y^{\frac{3}{2}}&=28. \end{aligned}\right\}$

28. $\left.\begin{aligned} &x^3+y^3=152, \\ &x^2y+xy^2=120. \end{aligned}\right\}$

29. $\left.\begin{aligned} &x^2+xy+y^2=a^2, \\ &x+(xy)^{\frac{1}{2}}+y=b. \end{aligned}\right\}$

30. $\left.\begin{aligned} &x+3(x+y)^{\frac{1}{2}}=18-y, \\ &x^2-y^2=9. \end{aligned}\right\}$

31. $\left.\begin{aligned} x^{\frac{3}{2}}+y^{\frac{2}{3}}&=3x, \\ x^{\frac{1}{2}}+y^{\frac{1}{3}}&=x. \end{aligned}\right\}$

32. $\left.\begin{aligned} &3x^2+4y^2=7xy, \\ &x^{\frac{3}{2}}-x^{\frac{1}{2}}y=\tfrac{2}{9}y^2. \end{aligned}\right\}$

33. $\left.\begin{aligned} &x^3+y^3=351, \\ &xy=14. \end{aligned}\right\}$

34. $\left.\begin{aligned} x^{\frac{3}{4}}+y^{\frac{3}{5}}&=126, \\ x^{\frac{1}{4}}+y^{\frac{1}{5}}&=6. \end{aligned}\right\}$

35. $\left.\begin{aligned} &x^4+y^4=257, \\ &x+y=5. \end{aligned}\right\}$

36. $\left.\begin{aligned} &x^4+y^4=337, \\ &xy=12. \end{aligned}\right\}$

37. $\left.\begin{aligned} &x^4-y^4=14560, \\ &x-y=8. \end{aligned}\right\}$

38. $\left.\begin{aligned} &x^4+y^4=641, \\ &xy(x^2+y^2)=290. \end{aligned}\right\}$

39. $\left.\begin{aligned} &x^5+y^5=a^5, \\ &x+y=b. \end{aligned}\right\}$

40. $\left.\begin{aligned} &x^2+z^2=3xz, \\ &x^5+z^5=2. \end{aligned}\right\}$

41. $\left.\begin{aligned} &x-y=1, \\ &(x^2+y^2)(x^3-y^3)=247. \end{aligned}\right\}$

42. $\left.\begin{aligned} &x^3y^2+xy^4=156, \\ &2x^3y^2-x^2y^3=144. \end{aligned}\right\}$

43. $\left.\begin{aligned} &\frac{y}{x}-\frac{81}{xy}=(2y+18)\frac{x^{\frac{1}{2}}}{y}, \\ &y+3x^{\frac{3}{2}}=9+3(x^3y)^{\frac{1}{2}}. \end{aligned}\right\}$

44. $\left.\begin{aligned} &\left(\frac{x}{y}\right)^{\frac{1}{2}}+\left(\frac{y}{x}\right)^{\frac{1}{2}}=\frac{61}{(xy)^{\frac{1}{2}}}+1, \\ &(x^3y)^{\frac{1}{4}}+(xy^3)^{\frac{1}{4}}=78. \end{aligned}\right\}$

45. $\left.\begin{aligned} &\frac{x}{y}-\frac{y}{x}=\frac{x+y}{x^2+y^2}, \\ &\frac{x^2}{y^2}-\frac{y^2}{x^2}=\frac{x-y}{y^2}. \end{aligned}\right\}$

46. $\left.\begin{aligned} &\frac{x^2}{y^2}+\frac{2x+y}{y^{\frac{1}{2}}}=20-\frac{y^2+x}{y}, \\ &x-4y+8=0. \end{aligned}\right\}$

47. $\left.\begin{aligned} &x^3+y^3+x^2y+xy^2=13, \\ &x^4y^2+x^2y^4=468. \end{aligned}\right\}$

48. $\left.\begin{aligned} &x^4+y^4=1+2xy+3x^2y^2, \\ &x^3+y^3=(1+x)(1+2y^2). \end{aligned}\right\}$

Ex. 42. Solve the equations—

49. $\left.\begin{aligned} x^2(b-y)&=ay(y-n), \\ y^2(a-x)&=bx(x-n). \end{aligned}\right\}$ 50. $\left.\begin{aligned} (x^2-xy+y^2)(x^2+y^2)&=221, \\ (x^2-xy+y^2)(x^2+xy+y^2)&=273. \end{aligned}\right\}$

51. $\left.\begin{aligned} x^4+y^2&=49+2x^2y, \\ x^4+y^4-x^2&=20+(2x^2-1)y^2. \end{aligned}\right\}$

52. $\left.\begin{aligned} x^4+5x^3y-3x^2y^2+5xy^3+y^4&=379, \\ x^4-3x^3y+5x^2y^2-3xy^3+y^4&=43. \end{aligned}\right\}$

53. $\left.\begin{aligned} (x^4+y^4)^2+x^2y^2(x^2-y^2)^2+x^2-y^2&=328, \\ x^2-y^2&=3. \end{aligned}\right\}$

54. $\left.\begin{aligned} x^2+y^2&=x+6, \\ x^3(2-x)+6y^2x-y^2(6x^2+y^2)&=x-5. \end{aligned}\right\}$

55. $\left.\begin{aligned} x(bc-xy)&=y(xy-ac), \\ xy(ay+bx-xy)&=abc(x+y-c). \end{aligned}\right\}$

56. $\left.\begin{aligned} x+(x^2-y^2)^{\frac{1}{2}}&=\frac{8}{y}\{(x+y)^{\frac{1}{2}}+(x-y)^{\frac{1}{2}}\} \\ (x+y)^{\frac{3}{2}}-(x-y)^{\frac{3}{2}}&=26. \end{aligned}\right\}$

57. $\left.\begin{aligned} (xy^2+x)^{\frac{1}{2}}+x^{\frac{1}{2}}&=y(x+9)^{\frac{1}{2}}+3y, \\ x(y+1)^2&=36(y^3+\tfrac{16}{9}). \end{aligned}\right\}$

58. $\left.\begin{aligned} \left(\frac{x}{a}\right)^{\alpha}\cdot\left(\frac{y}{b}\right)^{\beta}&=a^{\alpha\beta}, \\ \left(\frac{x}{b}\right)^{\beta}\cdot\left(\frac{y}{a}\right)^{\alpha}&=b^{\alpha\beta}. \end{aligned}\right\}$ 59. $\left.\begin{aligned} x^{\frac{\sqrt[4]{x}+\sqrt[4]{y}}{2}}&=y^{\frac{4}{3}}, \\ y^{\frac{\sqrt[4]{x}+\sqrt[4]{y}}{2}}&=x^{\frac{1}{3}}. \end{aligned}\right\}$

60. $\left.\begin{aligned} xy^2z^3&=108, \\ \frac{yz^2}{x}&=18, \\ \frac{z}{x^2y}&=\frac{3}{2}. \end{aligned}\right\}$ 61. $\left.\begin{aligned} xyz&=105, \\ \frac{x}{yz}&=\frac{3}{35}, \\ \frac{xy}{z}&=\frac{15}{7}. \end{aligned}\right\}$ 62. $\left.\begin{aligned} xyz&=231, \\ xyw&=420, \\ xzw&=660, \\ yzw&=1540. \end{aligned}\right\}$

63. $\left.\begin{aligned} x^2+xy+y^2&=37, \\ x^2+xz+z^2&=28, \\ y^2+yz+z^2&=19. \end{aligned}\right\}$ 64. $\left.\begin{aligned} x+y+z&=11, \\ x^2+y^2+z^2&=49, \\ yz&=3x(z-y). \end{aligned}\right\}$

65. $\left.\begin{aligned} x+y+z&=13, \\ x^2+y^2+z^2&=91, \\ y^2&=xz. \end{aligned}\right\}$ 66. $\left.\begin{aligned} x+y+z&=13, \\ x^2+y^2+z^2&=61, \\ x(y+z)&=2yz. \end{aligned}\right\}$

VII. Problems in Equations of the 2nd and higher Degrees.

Ex. 43.

1. Find two numbers, whose difference is two-ninths of the greater, and the difference of whose squares is 128.

2. The sum of two numbers is 16; and the quotient of the greater divided by the less is $2\frac{1}{4}$ times the quotient of the less by the greater: find them.

3. The difference of two numbers is 15, and half their product is equal to the cube of the less number: find them.

4. The product of two numbers is 24, and their sum multiplied by their difference is 20: find them.

5. The difference of the squares of two consecutive numbers is 17: find them.

6. The product of two numbers is 18 times their difference, and the sum of their squares is 117: find them.

7. What two numbers are those whose sum multiplied by the greater is 204; and whose difference multiplied by the less is 35?

8. There are two numbers such that the sum of the products of the first multiplied by 4 and of the second by 3 is 53; the difference of their squares is 15: find the numbers.

9. The product of two numbers added to their sum is 23; and 5 times their sum taken from the sum of their squares leaves 8: required the numbers.

10. Divide the number 14 into two parts, such that the sum of the quotients of the greater divided by the less, and of the less by the greater may be $2\frac{1}{12}$.

11. What two numbers are those whose sum added to the sum of their squares is 42, and whose product is 15?

12. A farmer bought some sheep for £72, and found that if he had received 6 more for the same money, he would have paid £1 less for each. How many sheep did he buy?

13. A and B distribute £60 each among a certain number of persons: A relieves 40 persons more than B does, and B gives to each 5*s.* more than A. How many persons did A and B respectively relieve?

14. A vintner sold 7 dozen of sherry and 12 dozen of claret for £50. He sold of sherry 3 dozen more for £10 than he did of claret for £6. Required the price of each.

15. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the

Ex. 43.

enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in 5 lines. Required the number of men.

16. The product of the sum and difference of the hypothenuse and a side of a right-angled triangle is 2; and 4 times the sum of the squares of the hypothenuse and this side is equal to 5 times the sum of these two lines: find the 3 sides of the triangle.

17. There are three numbers, the difference of whose differences is 5; their sum is 44, and continued product 1950: find the numbers.

18. Divide the number 26 into three such parts that their squares may have equal differences, and that the sum of those squares may be 300.

19. The sum of 4 numbers is 44; the sum of the products of the first and second, and third and fourth is 250; of the first and third, and second and fourth is 234; and of the first and fourth, and second and third 225: find them.

Inequalities.

Ex. 44.

1. Show that n^3+1 is $>n^2+n$.
2. If $x^2=a^2+b^2$, $y^2=c^2+d^2$; show that $xy>ac+bd$ or $ad+bc$.
3. If $a>b$; show that $a-b>(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2$.
4. If $x>y$; show that $x-y>\dfrac{x^4-y^4}{4x^3}$, but $<\dfrac{x^4-y^4}{4y^3}$.
5. Show that $xy^{-2}+x^{-2}y>x^{-1}+y^{-1}$; $x^2+y^2+z^2>xy+xz+yz$.
6. Show that $2(1+a^2+a^4)>3(a+a^3)$.
7. If $4b^2>a^2$; show that $x^2+b^2>ax$.
8. If $x>a$; show that $x^3+7ax^2>(x+a)^3$.
9. Show that $(a+b+c)^3>27abc$, but $<9(a^3+b^3+c^3)$.
10. Show that $abc>(a+b-c)(a+c-b)(b+c-a)$.
11. Show that $x^3+y^3+z^3>\frac{1}{2}(x^2y+xy^2+x^2z+xz^2+y^2z+yz^2)$.

RATIO, PROPORTION AND VARIATION.

Ex. 45.

1. Compare the ratios 7 : 8 and 10 : 11; 19 : 25 and 56 : 74.
2. Show that $a:b>ax:bx+h$; but $<ax:bx-h$.
3. Show that $a^3+b^3:a^2+b^2>a^2+b^2:a+b$.
4. Which is greater, $a+x:a-x$, or $a^2+x^2:a^2-x^2$?

Ex. 45.

5. Show that $a-x : a+x$ is $>$ or $< a^2-x^2 : a^2+x^2$ according as $a : x$ is a ratio of less or greater inequality.

6. Which is greater, $\dfrac{a+x}{a}$ or $\dfrac{4x}{a+x}$; $\dfrac{a^2-x^2}{a^3-x^3}$ or $\dfrac{a-x}{a^2-x^2}$?

7. Find the ratio compounded of $a+x : a-x$, $a^2+x^2 : (a+x)^2$, and $(a^2-x^2)^2 : a^4-x^4$.

8. If $a : b > c : d$; show that $a+c : b+d < a : b$, but $> c : d$.

9. If $a : b = c : d$; show that $7a+b : 3a+5b = 7c+d : 3c+5d$.

10. If a be the greatest of the 4 proportionals a, b, c, d; show that $a+d > b+c$; and that $a^n+d^n > b^n+c^n$.

11. If $a : b = c : d$; show that

$$a-c-(b-d)=\frac{(a-b)(a-c)}{a}; \quad \frac{1}{a}+\frac{1}{d}=\frac{1}{b}+\frac{1}{c}+\frac{(a-b)(a-c)}{abc}.$$

Find the equation between x and y,—

12. If $y \propto x$, and $y=21$, when $x=3$.

13. If $xy \propto x^2+y^2$, and $y=4$, when $x=3$.

14. If $y^2 \propto x^2-a^2$, and $y=\dfrac{b^2}{a}$, when $x=(a^2+b^2)^{\frac{1}{2}}$.

15. If $y^2 \propto x$, and $y=\pm 2a$, when $x=a$.

16. If $x \propto y$ and $y \propto z$; show that

$$(ax+by+cz) \propto \{h(xy)^{\frac{1}{2}}+k(xz)^{\frac{1}{2}}+l(yz)^{\frac{1}{2}}\}.$$

17. There are two vessels A and B each containing a mixture of water and wine, A in the ratio of 2 : 3, B in the ratio of 3 : 7. What quantity must be taken from each in order to form a third mixture which shall contain 5 gallons of water and 11 of wine?

18. The value of diamonds $\propto$ as the square of their weights, and the square of the value of rubies $\propto$ as the cube of their weights; a diamond of a carats is worth m times a ruby of b carats, and both together are worth £c: find the values of a diamond and ruby, each weighing x carats.

ARITHMETICAL PROGRESSION.

If S *be the sum of* n *terms of a series in Ar. Prog.,* a, l *the* 1*st and* n*th terms respectively,* d *the common difference*

$$l=a+(n-1)d; \qquad S=\{2a+(n-1)d\}\frac{n}{2}.$$

Ex. 46.

1. Find the 15th term of the series 3, 7, 11, &c.

Ex. 46.

2. Find the 11th term of the series $5, 1, -3$, &c.
3. Find the 20th term of the series $57, 54, 51$, &c.
4. Find the 8th term of the series $\frac{2}{3}, \frac{7}{12}, \frac{1}{2}$, &c.
5. Find the 19th term of the series $\frac{1}{5}, 1\frac{4}{25}, 2\frac{3}{25}$, &c.

Find the sum—

6. Of $1+3+5+$&c. to 20 terms.
7. Of $2+7+12+$&c. to 101 terms.
8. Of $12+7+2-3-$&c. to 9 terms.
9. Of $-5-1+3+$&c. to 12 terms.
10. Of $2+2\frac{1}{4}+2\frac{1}{2}+$&c. to 12 terms.
11. Of $13+12\frac{1}{3}+11\frac{2}{3}+$&c. to 40 terms.
12. Of $1+2\frac{2}{3}+4\frac{1}{3}+$&c. to 22 terms.
13. Of $2\frac{1}{3}+3\frac{5}{24}+4\frac{1}{12}+$&c. to 5 terms.
14. Of $1+\frac{6}{7}+\frac{5}{7}+$&c. to 15 terms.
15. Of $6+\frac{11}{2}+5+$&c. to 25 terms.
16. Of $17+\frac{49}{3}+15\frac{2}{3}+$&c. to 51 terms.
17. Of $-7-5\frac{2}{3}-4\frac{1}{3}-$&c. to 21 terms.
18. Of $\frac{1}{2}-\frac{2}{3}-\frac{11}{6}-$&c. to n terms.
19. Of $2\frac{2}{3}+2+\frac{4}{3}+$&c. to n terms.
20. Of $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+$&c. to n terms.
21. Of $(a+x)^2+(a^2+x^2)+(a-x)^2+$&c. to n terms.
22. Of $\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+\frac{5a-3b}{a+b}+$&c. to n terms.

How many terms of the series—

23. $7, 9, 11$, &c. amount to 40?
24. $19, 17, 15$, &c. amount to 91?
25. $54, 51, 48$, &c. amount to 513?
26. $\cdot034, \cdot0344, \cdot0348$, &c. amount to $2\cdot748$?
27. The first term of an AR. series is $\frac{10}{3}$, the common difference $\frac{13}{9}$, and the sum of the series 22; find the number of terms.

Ex. 46.

28. The first term of an AR. series is 5, the number of terms is 30, and their sum 1455; find the common difference.

29. The first term of an AR. series is 19, the last term $\frac{5}{2}$, and the nnmber of terms 12; find the common difference.

30. The sum of 11 terms of an AR. series is 22, and the common difference is $\frac{3}{5}$; find the first term.

31. The first term of an AR. series is 17, the last term $-12\frac{3}{8}$, and the sum $25\frac{7}{16}$; find the common difference.

32. Insert 3 AR. means between 117 and 477.

33. Insert 4 AR. means between 3 and 18.

34. Insert 4 AR. means between 2 and -18.

35. Insert 9 AR. means between 1 and -1.

36. Insert 7 AR. means between $-1\frac{1}{4}$ and $4\frac{3}{4}$.

37. There are n AR. means between 1 and 31, such that the 7th mean : $(n-1)$th mean$=5:9$; required n.

38. The sum of n AR. means between 1 and 19 : sum of the first $n-2$ of them :: 5 : 3; required n.

39. There are n AR. means between a and b, and between the pth and qth terms of these means there are r AR. means inserted; find the mth term of the last set.

40. The 5th and 9th terms of an AR. series are 13 and 25 respectively; what is the 7th term?

41. The nth term of an AR. series is $\frac{1}{6}(3n-1)$; find the first term, common difference, and the sum of n terms.

42. The sums of 2 AR. series each to n terms, are as $13-7n$: $1+3n$; find the ratio of their first terms, and also of their second terms.

43. In the series 1, 3, 5, &c. the sum of $2r$ terms : the sum of r terms :: x : 1; determine the value of x.

44. Find the ratio of the latter half of $2n$ terms of any AR. series, to the sum of $3n$ terms of the same series.

45. If m and n be the $(p+q)$th and $(p-q)$th terms respectively of an AR. series; find the pth and qth terms.

46. If a, b and c be the pth, qth and rth terms respectively of an AR. series; show that $a(q-r)+b(r-p)+c(p-q)=0$.

Ex. 46.

47. If $s_1, s_2, s_3, \ldots s_r$ be the sums of r AR. series, each continued to n terms; $1, 2, 3, \ldots r$ their first terms, and $1, 3, 5, \ldots (2r-1)$ their common differences respectively; required the sum of the series $s_1+s_2+s_3+\ldots+s_r$.

48. If $s_n, s_{n+1}, s_{n+2} \ldots$ denote the sums of $n, n+1, n+2, \ldots$ terms of an AR. series; find the sum of $s_n+s_{n+1}+s_{n+2}+\ldots$ to n terms.

49. In the two series $2, 5, 8, \ldots$ and $3, 7, 11, \ldots$ each continued to 100 terms; find how many terms are common to both series.

50. A debt can be discharged in a year by paying 1*s*. the first week, 3*s*. the second, 5*s*. the third, and so on; required the amount of the debt and the last payment.

51. From two towns 168 miles distant, A and B set out to meet each other; A went 3 miles the first day, 5 the second, 7 the third, and so on; B went 4 miles the first day, 6 the second, 8 the third, and so on: in how many days did they meet?

52. Find 4 numbers in AR. progression, such that the sum of the squares of the first and second be 29; and the sum of the squares of the third and fourth be 185.

53. Given P and Q the mth and nth terms of an AR. series; required the rth term.

Ex. 47. Find the sum—

1. Of $1^2+2^2+3^2+$&c. to 15 terms.
2. Of $1^2+3^2+5^2+$&c. to 21 terms.
3. Of $2^2+5^2+8^2+$&c. to n terms.
4. Of $1^3+2^3+3^3+$&c. to n terms.
5. Of $1.2+2.3+3.4+4.5+$&c. to 10 terms.
6. Of $1.3+3.5+5.7+7.9+$&c. to 12 terms.
7. Of $3.8+6.11+9.14+$&c. to n terms.
8. Of $2.5-4.7+6.9-$&c. to $2r$ terms.
9. Of the triangular numbers $1, 3, 6, 10, 15, \ldots$ to n terms.
10. Of the pyramidal numbers $1, 4, 10, 20, 35, \ldots$ to n terms.
11. Given $n^2+(n+1)^2+(n+2)^2+\ldots$ to 9 terms $=501$; required n.
12. Determine the AR. series of 11 terms, whose sum is 220, and the sum of their cubes 147400.

GEOMETRICAL PROGRESSION.

If S *be the sum of* n *terms of a series in Geom. Prog.*, a, z *the* 1*st and* n*th terms respectively*, r *the common ratio; then will*

$$z = ar^{n-1}, \qquad S = a\left(\frac{r^n - 1}{r - 1}\right).$$

If the series be convergent, in which case $r < 1$; *and be extended without limit so that* n *becomes* ∞; *then will the limiting sum*

$$\Sigma = \frac{a}{1 - r}.$$

Ex. 48.

1. Find the 5th term of the series 5, 10, 20, &c.
2. Find the 7th term of the series 54, 27, $13\frac{1}{2}$, &c.
3. Find the 6th term of the series $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$, &c.
4. Find the 5th term of the series 27, −45, 75, &c.
5. Find the 7th term of the series −21, 14, $-9\frac{1}{3}$, &c.

Find the sum—

6. Of $1 + 3 + 9 +$ &c. to 9 terms.
7. Of $8 + 20 + 50 +$ &c. to 7 terms.
8. Of $25 + 10 + 4 +$ &c. to 10 terms.
9. Of $9 - 6 + 4 -$ &c. to 9 terms.
10. Of $3 + 4\frac{1}{2} + 6\frac{3}{4} +$ &c. to 5 terms.
11. $\frac{1}{3} - \frac{1}{2} + \frac{3}{4} -$ &c. to 8 terms.
12. $\frac{4}{9} - \frac{1}{3} + \frac{1}{4} -$ &c. to 5 terms.

Find the sum of n terms—

13. Of $\frac{1}{3} - \frac{1}{4} + \frac{3}{16} -$ &c.
14. Of $\frac{1}{5} - \frac{2}{15} + \frac{4}{45} -$ &c.
15. Of $\frac{2}{5} - \left(\frac{2}{5}\right)^{\frac{1}{2}} + 1 -$ &c.
16. Of $3 + 9^{\frac{1}{3}} + 3^{\frac{1}{3}} +$ &c.

Find the limit of the sum of the following infinite series:

17. $4 + 2 + 1 +$ &c.
18. $9 - 6 + 4 -$ &c.
19. $\frac{2}{3} - \frac{1}{2} + \frac{3}{8} -$ &c.
20. $1 - \frac{2}{5} + \frac{4}{25} -$ &c.
21. $4 + 3 + \frac{9}{4} +$ &c.
22. $1 - \frac{2}{3} + \frac{4}{9} -$ &c.
23. $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} +$ &c.
24. $1 + 2 - \frac{1}{4} + \frac{1}{32} -$ &c.
25. $1 + \frac{2}{3} - \frac{1}{4} + \frac{3}{32} -$ &c.
26. $\frac{1}{3.2} - \frac{1}{3.2^3} + \frac{1}{3.2^5} -$ &c.
27. $1 - 2x + 2x^2 - 2x^3 +$ &c.
28. $\frac{f}{g} - \frac{f-g}{g^2}x + \frac{f-g}{g^3}x^2 -$ &c.

Ex. 48. Find the limit of the sum of the following infinite series:

29. $x-y+\dfrac{y^2}{x}-\dfrac{y^3}{x^2}+$&c. **30.** $\left(\dfrac{x}{y}\right)^{\frac{1}{2}}-\left(\dfrac{y}{x}\right)^{\frac{1}{2}}+\left(\dfrac{y}{x}\right)^{\frac{3}{2}}-$&c.

31. $\dfrac{a}{x}\left(\dfrac{3}{2}\right)^{\frac{1}{2}}+\left(\dfrac{a}{x}\right)^{\frac{1}{2}}+\left(\dfrac{2}{3}\right)^{\frac{1}{2}}+$&c.

32. Find the values of $\cdot 02\dot{7}$; $1\cdot\dot{1}\dot{4}5$; $\cdot 2\dot{1}5\dot{0}1$; $\cdot\dot{1}4285\dot{7}$.

33. Insert 2 Geom. means between 1 and 100; 4 between 2 and 64; 3 between $\frac{1}{2}$ and 128; and 4 between $\frac{2}{3}$ and $-5\frac{1}{16}$.

34. If P be the sum of the series formed by taking the 1st and every pth term of an infinite Geom. series, in which $a=1$, and r is <1; Q the sum of the series formed by taking the 1st and every qth term; prove that $P^q(Q-1)^p=Q^p(P-1)^q$.

35. If the Ar. mean between a and b be double the Geom.; find $\dfrac{a}{b}$.

36. The difference between 2 numbers is 48, and the Ar. mean exceeds the Geom. by 18; find the numbers.

37. If P and Q be the pth and qth terms of a Geom. series; find the nth term.

38. If m and n be the $(p+q)$th and $(p-q)$th terms of a Geom. series; find the pth and qth terms.

39. If $a, b, c, d \ldots$ be $n+1$ quantities in Geom. progression, show that the reciprocals of a^2-b^2, b^2-c^2, c^2-d^2, &c. are also in Geom. progression; and find the sum of the latter series.

40. If a, b, c, d be in Geom. progression; show that

$$(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2.$$

41. If a, b, c be in Geom. progression; show that

$$a^2+b^2+c^2>(a-b+c)^2.$$

42. If P be the continued product of n quantities in Geom. progression, S their sum, and S_1 the sum of their reciprocals; show that $P^2=\left(\dfrac{S}{S_1}\right)^n$.

43. The sum of £700 was divided among 4 persons, whose shares were in Geom. progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

44. Given S_1 and S_2 the sums of the even and odd terms respectively of a Geom. series of $2n$ terms. Of m Ar. means inserted between its pth and qth terms, required the rth mean.

Ex. 49. Find the sum—

1. Of $1-3x+5x^2-7x^3+$ &c. to infinity.
2. Of $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+$ &c. to infinity.
3. Of $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+$ &c. to infinity.
4. Of $1+5+13+29+$ &c. to n terms. [The rth term is $2^{r+1}-3$.]
5. Of $1+3+\ 7+15+$ &c. to n terms. [The rth term is 2^r-1.]
6. Of $3+6+11+20+$ &c. to n terms. [The rth term is 2^r+r.]
7. Of $2+4+\frac{14}{3}+\frac{44}{9}+$ &c. to n terms. [The rth term is $5-3^{-r+2}$.]
8. Of $1.1+2.3+4.5+8.7+$ &c. to n terms.
9. Of $1.2x+2.3x^2+3.4x^3+$ &c. to infinity, and to n terms.
10. Of $1.2x+2.4x^2+3.8x^3+$ &c. to infinity.
11. Of $1.3x+4.9x^2+7.27x^3+$ &c. to infinity.
12. Find the 9th term of the series 4, 7, 10, 16, 28, &c.

HARMONICAL PROGRESSION.

If a, b, c, d, *&c. be in Harmonical progression,*

$$a : c = a-b : b-c; \qquad b : d = b-c : c-d;$$

and $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, *&c. are in Ar. progression.*

Ex. 50.

1. Continue to 3 terms each way the series 2, 3, 6; $1\frac{1}{2}$, $2\frac{1}{7}$, $3\frac{3}{4}$; and 1, $1\frac{1}{7}$, $1\frac{1}{3}$.

2. Insert 2 Harm. means between 2 and 4; and 4 between 2 and 12.

3. Insert 6 Harm. means between 9 and 3; and 3 between 1 and 20.

4. Insert n Harm. means between x and y.

5. Find a fourth Harm. proportional to 12, 6, 4.

6. Given x and y the 1st and 2nd terms of a Harm. progression; continue the series, and write down the nth term.

7. Given M and N the mth and nth terms of a Harm. progression; find the $(m+n)$th term.

8. If a, b, c be the pth, qth, and rth terms respectively of a Harm. series; show that $(p-q)ab+(r-p)ac+(q-r)bc=0$.

9. If a, b, c be in Harm. progression, show that $a^2+c^2>2b^2$; and if n be a positive integer, $a^n+c^n>2b^n$.

Ex. 50.

10. The sum of 3 numbers in Harm. progression is 26, and the product of the extremes exceeds the square of the mean, by the mean; find the numbers.

11. The sum of 3 numbers in Harm. progression is 37, and the sum of their squares is 469; find the numbers.

12. The sum of 3 numbers in Harm. progression is 11, and their continued product is 36; find the numbers.

13. Compare the lengths of the sides a, b, and c of a right-angled triangle, c being the hypothenuse, when the squares described upon them are in Harm. progression.

14. Show that the Ar. Geom. and Harm. means between a and b are in continued proportion.

15. If y be the Harm. mean between x and z, and x and z be the Ar. and Geom. means respectively between a and b; express y in terms of a and b.

16. There are 4 nos. of which the first 3 are in Ar. progression, the last 3 in Harm.; show that the 1st : 2nd : : 3rd : 4th.

17. If $a^x = b^y = c^z =$ &c., and a, b, c, &c. be in Geom. progression, then will x, y, z, &c. be in Harm. progression.

PILES OF BALLS AND SHELLS.

If n *be the number of balls in a side of the base row;*

$\frac{1}{6}n(n+1)(n+2)$ *is the number of balls in the triangular pile;*

$\frac{1}{6}n(n+1)(2n+1)$ . . . . *square* . .;

$\frac{1}{6}n(n+1)(2n+1+3d)$ . . . . *rectangular* . .,

where $n+d$ *is the number of balls in the longer side of the base.*

Ex. 51. Find the number of balls in—

1. A triangular pile, each side of the base having 36 balls.

2. A square pile, each side of the base having 32 balls.

3. A rectangular pile, the length and breadth of the base containing 52 and 34 balls respectively.

4. An incomplete triangular pile, a side of the base course having 25 balls, and a side of the top 13.

5. An incomplete triangular pile of 15 courses, having 38 balls in a side of the base.

Ex. 51. Find the number of balls in—

6. An incomplete square pile, a side of the base course having 44 balls, and a side of the top 22.

7. An incomplete square pile of 30 courses, having 65 balls in each side of the base.

8. An incomplete rectangular pile of 18 courses, having 56 balls in the length, and 38 in the breadth of the base.

9. An incomplete rectangular pile of 25 courses, having 100 balls in the length of the base, and 35 in the breadth of the top.

10. An incomplete rectangular pile, having 12 and 26 balls respectively in the shorter sides of its top course and base, and 45 balls in the longer side of its base.

11. An incomplete square pile, the upper course consisting of 529 balls, and the base of 5184.

12. The number of balls in a complete rectangular pile of 20 courses is 6440; how many balls are in its base?

13. The number of balls in the shorter side of the base of a complete rectangular pile is 15; how many must there be in the other side that the pile may contain 4960 balls?

14. The number of balls in a triangular pile is to the number in a square pile, having the same number of balls in the side of the base, as 6 to 11; required the number in each pile.

PERMUTATIONS AND COMBINATIONS.

1. *The number of permutations of* n *different things, taking* r *of them at a time, is* $n(n-1)(n-2)\ldots(n-r+1)$.

2. *The number of permutations of* n *things, when* p *of them are of one sort,* q *of another,* r *of a third, and so on; all taken together is*

$$\frac{1.2.3\ldots n}{(1.2.3\ldots p)(1.2.3\ldots q)(1.2.3\ldots r)(\&c.)}.$$

3. *The number of combinations of* n *different things, taking* r *of them at a time, is,* $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}$.

Ex. 52.

1. How many changes may be rung with 4 bells out of 7; and how many with the whole peal?

2. How many different signals may be formed by means of 12 different flags which can be hoisted 4 at a time above each other?

3. In how many different ways may 8 persons be seated at table?

Ex. 52.

4. In how many different ways may the letters of the continued product a^5ce^4f be written?

5. Find the number of permutations that can be formed out of the letters of the words *Mississippi, Museum, Meeanee, Nusserree.*

6. The No. of Perm. of n things, 5 together, is 20 times the No. 3 together: find n.

7. The No. of things : No. of Perm. 3 together : : 1 : 72; find the No. of things.

8. What is the No. of things, when the No. of Perm. is 5040?

9. There are 6 letters of which a certain No. are a's; and 120 words can be formed of them: how many a's are there?

10. If $p_2, p_3, \ldots p_n$ be the Nos. of Perm. that can be formed out of n quantities taken 2, 3, &c. n, together respectively; show that $p_2 p_3 \ldots p_n = p_3 p_n \{(p_3 - p_2)(p_4 - p_3)(p_5 - p_4) \ldots (p_{n-1} - p_{n-2})\}$.

11. Find the No. of Combs. of 10 things, 4 together; and also 6 together.

12. Find the No. of Combs. of 10 letters $a, b, c \ldots$, 5 together; in how many of the Combs. will a and b occur together?

13. Find the No. of Combs. of 12 letters $a, b, c \ldots$, 4 together; in how many of the Combs. will a, b and d occur together?

14. On how many nights may a different patrol of 5 men be draughted from a corps of 36? on how many of these would any one man be taken?

15. The No. of Combs. of n things, 4 together, is $7\frac{1}{2}$ times the No. of Combs., 2 together; find n.

16. The No. of Combs. of n things, 5 together, is 3·6 times the No. of Combs., 3 together; find n.

17. The No. of Combs. of $n+2$ things, 3 together, is 11 times the No. of Combs. of $\frac{2n}{3}$ things, 2 together; find n.

18. At a game of cards, 3 being dealt to each person, any one can have 425 times as many hands as there are cards in the pack: required the No. of cards.

19. How many Combs. can be formed out of 7 things taken 1, 2, 3, 4, 5, 6, 7 together respectively?

20. The total No. of Combs. of $2n$ things is 65 times the total No. of Combs. of n things; find n.

21. How many different sums can be formed with the following

Ex. 52.

coins: a farthing, a penny, a sixpence, a shilling, a half-crown, a crown, a half-sovereign, and a sovereign?

22. Out of 17 consonants and 5 vowels, how many words can be formed, each consisting of 2 vowels and 3 consonants?

23. How many words of 6 letters may be formed out of 24 letters of the alphabet, with 2 of the 5 vowels in each word?

24. The No. of Perms. of n things, 3 together, is 6 times the No. of Combs. 4 together; find n.

25. The No. of Perms. of n things taken r together is equal to 10 times the No. when taken $r-1$ together; and the No. of Combs. of n things taken r together is to the No. when taken $r-1$ together as 5 : 3; required the values of n and r.

26. If, generally, $C_m^{(p)}$ denote the No. of Combs. of m things taken p together; show that $C_{n+1}^{(r)} = C_n^{(r)} + C_n^{(r-1)}$.

27. A person wishes to make up as many *different* parties as he can out of 20 friends, each party consisting of the same number; how many should he invite at a time?

28. When the No. of Combs. of $2n$ things taken r together is the greatest possible; required r.

29. There are 4 regular polyhedrons marked, each face with a different symbol, and the numbers of their faces are 4, 6, 8, 12 respectively; taking all of them together, how many different throws are possible?

30. Find the No. of *different* Combs. of n things, of which p are of one sort, q of another, r of a third, and so on, when taken 1, 2, 3, &c. n together severally.

BINOMIAL AND MULTINOMIAL THEOREMS.

1. *In the expansion of* $(a+x)^n$,
the $(r+1)$*th term is* $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}a^{n-r}x^r$.

2. *In the expansion of* $(a+bx+cx^2+\ldots+kx^t)^n$,
the term involving x^m *is* $\dfrac{1.2.3\ldots n}{(1.2.3..p)(1.2.3..q)(1.2.3..r)\&c.}a^p.b^q.c^r.\&c.x^{q+2r+\ldots}$
where $p+q+r+\ldots=n$, *and* $q+2r+\ldots=m$.

Ex. 53.

1. Expand $(1+x)^7$; $(1+2x)^5$; $\left(1+\dfrac{x}{2}\right)^6$; $\left(1+\dfrac{3x}{4}\right)^4$.

Ex. 53.

2. Expand $(1-3x)^6$; $\left(1-\frac{x}{2}\right)^5$; $\left(1-\frac{x}{4}\right)^{-4}$; $\left(1+\frac{2}{3}x^2\right)^{-5}$.

3. Expand $(a+x)^9$; $(a-x)^7$; $(2x-3y)^4$; $\left(5-\frac{x}{6}\right)^6$.

4. Expand $(a^2-2x)^{-5}$; $(3a^{-1}-2x)^{-4}$; $(c-x)^{-t}$; $(a+h)^{-s}$.

5. Expand $(1+2x)^{\frac{1}{2}}$; $(1-x)^{\frac{5}{6}}$; $(a+\frac{4}{3}x)^{\frac{3}{4}}$; $(\frac{2}{3}x-\frac{3}{2}y)^{\frac{2}{3}}$.

6. Expand $(1-x)^{-\frac{1}{2}}$; $(1-x^2)^{-\frac{3}{2}}$; $(a^2-x^2)^{-\frac{1}{3}}$; $(x^5+z^5)^{-\frac{2}{5}}$.

7. Expand $\dfrac{1}{a+a^{\frac{2}{3}}x^{\frac{1}{3}}}$; $\dfrac{1}{(a^{\frac{1}{3}}-x^{\frac{1}{3}})^6}$; $\dfrac{3a}{(ax-x^2)^{\frac{1}{3}}}$.

8. Expand $(h+k\sqrt{-1})^7$; $(b-y\sqrt{-1})^8$; $(-a^3+x\sqrt{-1})^n$.

9. Expand $(1+x+x^2)^5$; $(1-2x+x^2)^3$; $(a-2b+3c)^4$.

10. Expand $(1+x+x^2+\ldots \text{in inf.})^2$; $(1+x+x^2+\ldots \text{in inf.})^{\frac{1}{2}}$.

Find the coefficient—

11. Of x^5 in $(a-x)^9$.

12. Of x^9 in $(5a^3-4x^3)^7$.

13. Of x^{12} in $(a^5-b^3x^2)^{\frac{5}{2}}$.

14. Of x^4 in $(1+3x-x^2)^5$.

15. Of $a^2b^2c^3$ in $(a+b+c)^7$.

16. Of x^5 in $(a-bx-cx^2)^{\frac{m}{n}}$.

17. Of $a^{n-5}b^2c^3$ in $(a+b+c)^n$.

18. Of x^r in $(3a+2x)^{-\frac{1}{5}}$.

19. Of x^5 in $(a+bx+cx^2+dx^3)^6$.

20. Of x^8 in $(2+4x-3x^2+x^3)^7$.

21. Of x^5 in $(1+2x+3x^2+\ldots \text{in inf.})^6$.

22. Of x^r in $(1+2x+3x^2+\ldots \text{in inf.})^2$.

23. Of $b^2c^3e^4f$ in $(a+2b+3c+4d+5e+6f)^{10}$.

Find in the following binomials or multinomials expanded—

24. The 6th term of $(4a^2cx-3c^{\frac{2}{3}}y^{\frac{1}{3}})^{\frac{1}{2}}$; and of $(ax-bx^2\sqrt{-1})^{\frac{3}{5}}$.

25. The 5th term of $(3x-2y)^{-10}$; and of $(-s+t\sqrt{-1})^{\frac{5}{7}}$.

26. The $(r+1)$th term of $\{xy-(3yz)^{\frac{1}{2}}\}^{\frac{5}{4}}$.

27. The greatest term of $(1+\frac{5}{6})^{\frac{3}{2}}$.

Ex. 53. Find in the following binomials or multinomials expanded,—

28. The middle term of $(1+x)^{2n}$; and of $(1+x+x^2)^{12}$.

29. The middle term of $(2-5x-7x^2+x^3+3x^4)^5$.

30. The No. of terms in $(a+b+c)^9$; and in $(a+bx+cx^2+dx^3)^4$.

31. Show that

$$\left(x+\frac{1}{x}\right)^{2n}=\left(x^{2n}+\frac{1}{x^{2n}}\right)+2n\left(x^{2n-2}+\frac{1}{x^{2n-2}}\right)+\frac{2n(2n-1)}{1.2}\left(x^{2n-4}+\frac{1}{x^{2n-4}}\right)$$
$$+\&c.+\frac{1.3.5...(2n-1)}{1.2.3...n}2^n.$$

$$\left(x-\frac{1}{x}\right)^{2n+1}=\left(x^{2n+1}-\frac{1}{x^{2n+1}}\right)-(2n+1)\left(x^{2n-1}-\frac{1}{x^{2n-1}}\right)$$
$$+\frac{(2n+1)2n}{1.2}\left(x^{2n-3}-\frac{1}{x^{2n-3}}\right)-\&c.+(-1)^n\frac{1.3.5...(2n+1)}{1.2.3...(n+1)}2^n\left(x-\frac{1}{x}\right).$$

$$\left(\frac{1+2x}{1+x}\right)^n=1+n\left(\frac{x}{1+2x}\right)+\frac{n(n+1)}{1.2}\left(\frac{x}{1+2x}\right)^2+\&c.$$

$$(a-x)^n=a^n\left\{1-n\left(\frac{x}{a-x}\right)+\frac{n(n+1)}{1.2}\left(\frac{x}{a-x}\right)^2-\&c.\right\}$$

$$\left(1+\frac{x}{n}\right)^n=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+...;\text{ when } n \text{ is infinite.}$$

Find the values of the following infinite series—

32. $1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{2^3}+...$

33. $1-2n+3n\cdot\frac{n+1}{2}-4n\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}+...$

34. $1+\frac{1}{2}n+\frac{1}{3}\cdot\frac{n(n-1)}{1.2}+\frac{1}{4}\cdot\frac{n(n-1)(n-2)}{1.2.3}+...$

35. $a-(a+b)n+(a+2b)\frac{n(n-1)}{1.2}-(a+3b)\frac{n(n-1)(n-2)}{1.2.3}+...$

36. If the coefficients of the $(r+1)$th and $(r+3)$th terms of $(1+x)^n$ are equal, n being a positive integer; find r.

37. The coefficients of x in the 5th and 7th terms of $(1+2x)^n$ are 1120 and 1792 respectively; find n.

38. The coefficients of x in the 3rd and 5th terms of $(1-x)^n$ are $\frac{14}{9}$ and $-\frac{7}{243}$ respectively; find n.

Ex. 53.

39. If generally n_r be the coefficient of the $(r+1)$th term of $(1+x)^n$, show that $(n+p)_r = n_r + n_{r-1}p_1 + n_{r-2}p_2 + \&c. + n_1 p_{r-1} + p_r$.

40. If generally m_r be the coefficient of the $(r+1)$th term of $(1-x)^{-m}$, show that $m_r + (m+1)_{r-1} = (m+1)_r$.

41. Find the sum of the squares of the coefficients in the expansion of $(1+x)^n$, when n is a positive integer.

N.B. Equate the coeffts. of x^n in $(1+x)^n.(x+1)^n$ and in $(1+x)^{2n}$.

42. Find the sum of the products of every two consecutive coefficients in the expansion of $(1+x)^n$, n being a positive integer.

N.B. Equate the coeffts. of x^{n-1} or x^{n+1} in $(1+x)^n.(x+1)^n$ and in $(1+x)^{2n}$.

43. If a, b, c, d be any consecutive coefficients of an expanded binomial, show that $(bc+ad)(b-c) = 2(ac^2 - b^2d)$.

44. If s = sum of two quantities, p = their product, and q = the quotient; show that $p^2 = s^4\left(q^2 - 4q^3 + \frac{4.5}{1.2}q^4 - \frac{4.5.6}{1.2.3}q^5 + \&c.\right)$.

45. If s = sum of the squares of any two quantities, $p = 2 \times$ product, and P = the pth power of the sum; show that

$$P.P^{\frac{1}{2}}.P^{\frac{1}{4}}.P^{\frac{1}{8}} \&c. \text{ in inf.} = s^p\left\{1 + p\left(\frac{p}{s}\right) + \frac{p(p-1)}{1.2}\left(\frac{p}{s}\right)^2 + \frac{p(p-1)(p-2)}{1.2.3}\left(\frac{p}{s}\right)^3 + \right.$$

INDETERMINATE COEFFICIENTS.

Ex. 54. Reduce into simple or partial fractions—

1. $\dfrac{6x^2-4x-6}{(x-1)(x-2)(x-3)}$. 2. $\dfrac{x^2+hx+k}{(x-a)(x-b)(x-c)}$.

3. $\dfrac{x^2}{(x^2-1)(x-2)}$. 4. $\dfrac{1}{x^4-a^4}$.

5. $\dfrac{1+5x+3x^2}{(1+x)^2(1+2x)^2}$. 6. $\dfrac{x^2+x+1}{x^3(x^2+1)^2}$.

7. $\dfrac{x-1}{x^3+8x^2+21x+18}$. 8. $\dfrac{3x^2-8x+16}{2x^3-14x^2+16x+20}$.

Expand in a series of ascending powers of x—

9. $\dfrac{3+2x}{5+7x}$. 10. $\dfrac{1+2x}{1-x-x^2}$. 11. $\dfrac{x-ax^3+bx^5}{1-cx^2+dx^4}$.

Ex. 54. Expand in a series of ascending powers of x—

12. $\dfrac{3x-2}{(x-1)(x-2)(x-3)}$; and find the coefficient of x^n.

REVERSION OF SERIES.

Ex. 55. Find the value of x in an infinite series, in terms of y—

1. When $y=1-2x+3x^2$.
2. When $y=a+bx+cx^2$.
3. When $y=1+x-2x^2+x^3$.
4. When $y=a+bx+cx^2+dx^3$.
5. When $y=1+2x+4x^2+8x^3+\ldots$
6. When $y=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots$
7. When $y=1+x+\dfrac{x^2}{1.2}+\dfrac{x^3}{1.2.3}+\dfrac{x^4}{1.2.3.4}+\ldots$
8. When $y=x-\dfrac{x^3}{1.2.3}+\dfrac{x^5}{1.2.3.4.5}-\dfrac{x^7}{1.2.3.4.5.6.7}+\ldots$
9. When $y=x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5}+\dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7}+\ldots$
10. When $y=ax+bx^2+cx^3+\ldots$
11. When $x^3-3x+y=0$.
12. When $n(x^3-1)-xy=0$.
13. When $x^3-axy-b^3=0$.

SUMMATION OF SERIES.

Ex. 56. Find, by the method of Indeterminate Coefficients, the sum—

1. Of $1^2+2^2+3^2+4^2+$&c. to 11 terms.
2. Of $1^2+4^2+7^2+10^2+$&c. to n terms.
3. Of $1.2+2.3+3.4+$&c. to 10 terms.
4. Of $1.2+3.4+5.6+$&c. to n terms.
5. Of $1.2^2+2.3^2+3.4^2+$&c. to n terms.
6. Of $1.2.3+2.3.4+3.4.5+$&c. to n terms.
7. Of $1^3+2^3+3^3+$&c. to 20 terms.
8. Of $1^3+3^3+5^3+$&c. to n terms.
9. Of 15 terms of a series whose nth term is $(2n-1)(3n+1)$.

Ex. 57. Find, by the method of Subtraction, the sum—

1. Of $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$ to n terms, and to infinity.

2. Of $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots$ to n terms, and to infinity.

3. Of $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots$ to n terms, and to infinity.

4. Of $\frac{1}{3.8}+\frac{1}{8.13}+\frac{1}{13.18}+\ldots$ to n terms, and to infinity.

5. Of $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots$ to n terms, and to infinity.

6. Of $\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+\ldots$ to n terms, and to infinity.

7. Of $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\ldots$ to n terms, and to infinity.

8. Of $\frac{2}{1.3.5.7}+\frac{3}{3.5.7.9}+\frac{4}{5.7.9.11}+\ldots$ to n terms, and to infinity.

Ex. 58. Find, by the method of Multiplication, the sum—

1. Of $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots$ to n terms, and to infinity.

2. Of $\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots$ to n terms, and to infinity.

3. Of $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots$ to n terms, and to infinity.

4. Of $\frac{3}{1.2}.\frac{1}{2}+\frac{4}{2.3}.\frac{1}{2^2}+\frac{5}{3.4}.\frac{1}{2^3}+\ldots$ to n terms, and to infinity.

5. Of $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\ldots$ to n terms, and to infinity.

6. Of $\frac{19}{1.2.3}.\frac{1}{4}+\frac{28}{2.3.4}.\frac{1}{8}+\frac{39}{3.4.5}.\frac{1}{16}+\frac{52}{4.5.6}.\frac{1}{32}+\ldots$ to n terms, and to infinity.

Ex. 59. In the following *Recurring* series find the sum—

1. Of $1+2x+3x^2+4x^3+\ldots$ to n terms.
2. Of $1+4x+7x^2+10x^3+\ldots$ to n terms.

Ex. 59. In the following *Recurring* series find the sum—

3. Of $1^2+2^2x+3^2x^2+4^2x^3+\ldots$ to n terms.

4. Of $1^2+3^2x+5^2x^2+7^2x^3+\ldots$ to n terms.

5. Resolve $1-\frac{7}{3}x+\frac{17}{9}x^2-\frac{55}{27}x^3+\ldots$ into its constituent Geom. progressions.

Ex. 60. Find, by the method of Differences—

1. The 12th term of 1, 5, 15, 35, 70, 126, &c.
2. The 9th term of 2.5.7, 4.7.9, 6.9.11, &c.
3. The sum of $1.2+2.3+3.4+4.5+\ldots$ to n terms.
4. The sum of $1.2.3+2.3.4+3.4.5+\ldots$ to n terms.
5. The sum of $1.2.5+3.4.7+5.6.9+\ldots$ to n terms.
6. The sum of $2.5.6+4.7.8+6.9.10+\ldots$ to n terms.
7. The sum of $1^3+2^3+3^3+4^3+\ldots$ to n terms.
8. The sum of $1.1^3+3.2^3+5.3^3+7.4^3+\ldots$ to n terms.
9. The sum of $3+11+31+69+131+$&c. to 20 terms.
10. The sum of $1+11+19+30+48+76+$&c. to 20 terms.
11. The sum of $1+6+21+56+126+252+462+$&c. to 15 terms.
12. The sum of 25 terms of a series whose nth term is $n^2(3n-2)$.
13. The sum of 20 terms of each of the 5 orders of figurate Nos.

Ex. 61. INTERPOLATION OF SERIES.

1. Find the 5th term of the series of which the 6th differences vanish, and the 1st, 2nd, 3rd, 4th, 6th, 7th terms are 11, 18, 30, 50, 132, 209.

2. Find the 2nd term of the series of which the 4th differences vanish, the 1st, 3rd, 4th and 5th terms being 3, 15, 30, 55; and continue the series to 10 terms.

3. Given the square roots of 19, 20, 21, 23; find that of 22.

4. Given the cube roots of 121, 122, 124, 125; find that of 123.

5. Insert three equidistant terms between every two consecutive terms of 1, 4, 10, 20, 35, &c.

Ex. 62. CHANCES OR PROBABILITIES.

1. What is the probability of throwing an ace in the first only of two successive throws?

Ex. 62.

2. What is the probability of drawing the 4 aces from a pack of cards in 4 successive trials?

3. There are 4 white balls and 3 black placed at random in a line; find the probability of the extreme balls being both black.

4. Of two urns, one contains 12 balls, the other 7, which are marked with letters, both beginning with *a*, *b*, *c*, &c. If a ball be drawn from both urns, what is the probability that the two will have the same letter-mark?

5. In a lottery containing black and white balls, where it is as likely that each drawing will yield a black as a white ball, what is the probability that in 11 trials, 11 white balls will be drawn?

6. In the game of '*heads* and *tails*,' what is the probability that heads will come up 3 times exactly in 7 trials?

7. In 5 throws with a single die, find the chance that an ace will be thrown at least twice.

8. An urn contains 7 white balls, 12 red and 10 black. If two of these balls be drawn, what is the probability that one will be white and the other red?

9. From a bag containing 2 guineas, 3 sovereigns and 5 shillings, a person is allowed to draw 3 of them indiscriminately; what is the value of his expectation?

10. An urn contains 3 white, 4 black and 5 red balls; what is the chance of drawing 1 white, 2 black and 3 red balls in six successive trials?

11. There are 3 balls in a bag, one of them is white, another black, and the third, it is equally probable, is white or black; determine the chance of drawing 2 white balls, if 2 be drawn.

12. If the House of Commons consist of m Tories and n Whigs; find the probability that a committee of $p+q$ members selected by ballot will consist of p Tories and q Whigs.

13. Ten balls are drawn at random from an urn containing 20 white balls and 5 black; what is the probability that 3, and no more, black balls will then be drawn?

14. At a game of whist, what is the chance of dealing one ace and no more to a specified person? And what is the chance of dealing one ace to each person?

15. A throws 6 dice, B throws 12, C throws 18. Compare the chances of A's throwing one six, B two sixes, and C three sixes.

16. There are two urns, in one of which are 5 white balls and

Ex. 62.

4 black; in the other, 3 white and 2 black: what is the probability that if one ball be drawn it will be white?

17. What is the chance of throwing an ace once only in three trials?

18. A die is thrown time after time; in how many times have we an even chance of throwing an ace?

19. If there be 12 bags, each containing 7 white balls and 2 black; and a ball be drawn from each bag successively: determine beforehand what is the most probable number of white balls that will be drawn.

20. What is the probability of throwing the point 16 with four common dice?

SCALES OF NOTATION.

N.B. The letters *t*, *e* are used in the following examples to denote 10, 11 respectively.

Ex. 63.

1. Express the denary No. 5381 in the ternary and nonary scales.

2. Express the common Nos. 34705 and 790158 in the septenary scale.

3. Express the quinary No. 34402 in the quaternary scale.

4. Express the common Nos. 6587 and 3907 in the duodenary scale; and then find their product.

5. Express the undenary Nos. 8978 and 3256 in the duodenary scale; and then find their product.

6. Multiply 24305 by 34120 in the senary scale.

7. Multiply 59*t*4 by 7906 in the undenary scale.

8. Divide 14332216 by 6541 in the septenary scale.

9. Divide 95088918 by *tt*4 in the duodenary scale.

10. Divide *ttet*1222 by *teet* in the duodenary scale.

11. Extract the square roots of 25400544 in the senary scale, and of 32*e*75721 in the duodenary.

12. In what scale is 40501 equivalent to the denary No. 5365?

13. In what scale is 147 equivalent to the denary No. 124?

14. The denary No. 4954 expressed in another scale is 20305; find the radix of that scale.

15. Find a fraction in the denary scale equivalent to the senary number $45{\cdot}2\dot{5}3\dot{4}$, &c.

Ex. 63.

16. Which of the weights 1 lb., 2 lb., 4 lb., 8 lb., &c. must be selected to weigh 1719 lb.?

17. Which of the weights 1 lb., 3 lb., 9 lb., &c. must be selected to weigh 304 lb.?

18. If N, N′ be any 2 numbers in the denary scale, composed of the same digits differently arranged; prove that $N \sim N'$ is divisible by 9.

19. Any number consisting of an even number of digits in a system whose radix is r is divisible by $r+1$, if the digits equidistant from each end are the same.

LOGARITHMS.

Ex. 64. Find, by logarithms, the product—

1. Of $24{\cdot}13 \times 6{\cdot}052$.
2. Of $49{\cdot}51 \times 283{\cdot}605$.
3. Of $5{\cdot}281925 \times 4{\cdot}375921$.
4. Of $20{\cdot}192248 \times 634{\cdot}47$.
5. Of $864665{\cdot}2 \times 8{\cdot}097466$.
6. Of $2487492 \times {\cdot}006988964$.
7. Of ${\cdot}007461 \times {\cdot}3351767$.
8. Of ${\cdot}0700379 \times {\cdot}0086752$.
9. Of ${\cdot}034632 \times {\cdot}397302$.
10. Of ${\cdot}00087214 \times {\cdot}001963$.
11. Of $4{\cdot}002 \times 608{\cdot}27 \times {\cdot}0425839$.
12. Of $63{\cdot}87009 \times 25603{\cdot}01 \times {\cdot}000725$.
13. Of $4{\cdot}697 \times 3{\cdot}2157 \times {\cdot}9483 \times {\cdot}0305$.
14. Of $4100 \times 7{\cdot}319 \times {\cdot}03 \times 439257 \times {\cdot}0000045879$.

Ex. 65. Find, by logarithms, the quotient—

1. Of $35274 \div 5678$.
2. Of $48{\cdot}25 \div 634{\cdot}87$.
3. Of $11 \div {\cdot}3929$.
4. Of ${\cdot}9649 \div 35{\cdot}0583$.
5. Of ${\cdot}26439 \div {\cdot}28629$.
6. Of ${\cdot}07425 \div {\cdot}008352$.
7. Of $5{\cdot}764231 \div {\cdot}00158$.
8. Of ${\cdot}84750 \div 14{\cdot}36009$.
9. Of ${\cdot}0697565 \div {\cdot}9975641$.
10. Of ${\cdot}001048869 \div {\cdot}00471698$.

Ex. 66. Find, by logarithms,—

1. The square of $35{\cdot}7924$.
2. The cube of $31{\cdot}097$.
3. The cube of $5{\cdot}0008562$.
4. The 4th power of ${\cdot}05632$.
5. The 5th power of ${\cdot}948008$.
6. The 13th power of $1{\cdot}0975$.
7. The 19th power of $1{\cdot}001786$.
8. The 70th power of $1{\cdot}0009$.
9. The 150th power of $1{\cdot}0035$.
10. The 11th power of ${\cdot}809$.

Ex. 67. Find, by logarithms,—

1. The square Root of $3{\cdot}621409$.
2. The cube Root of 3852.
3. The cube Rt. of $199586{\cdot}251$.
4. The 5th Rt. of $24871{\cdot}53$.
5. The cube Rt. of $\cdot 00052653$.
6. The 2nd Rt. of $\cdot 00780908$.
7. The 5th Rt. of $\cdot 0856329$.
8. The 11th Rt. of $7854{\cdot}39$.
9. The 19th Rt. of $\cdot 00123456$.
10. The 20th Rt. of 5.
11. The 365th Rt. of $1{\cdot}045$.
12. The $3{\cdot}5$th Rt. of $\cdot 7289$.
13. The $\cdot 8$th Rt. of $\cdot 08$.
14. The $\cdot 065$th Rt. of $1{\cdot}6235$.

Ex. 68. Find a fourth proportional—

1. To 8352, $3{\cdot}69$, $30{\cdot}57$.
2. To $357{\cdot}109$, $5000{\cdot}8$, $\cdot 031$.
3. To $\frac{197}{235}$, $\frac{641}{863}$, $\cdot 099$.
4. To the cube Rts. of $\cdot 21$, $\cdot 23$, $\cdot 25$.
5. To $(\cdot 00058309)^{\frac{1}{5}}$, $(\cdot 2839)^{3}$, $(\cdot 018 \div 25)^{\frac{1}{7}}$.

Find a third proportional—

6. To $\cdot 00709$, $\cdot 1508$.
7. To $5{\cdot}2\dot{4}\dot{1}$, $9{\cdot}5\dot{3}0\dot{8}$.
8. To $(\cdot 948)^{5}$, $(\cdot 00052653)^{\frac{1}{3}}$.

Find a mean proportional—

9. To $(\cdot 03)^{\frac{1}{3}}$, $(\cdot 529807)^{5}$.
10. To $(\cdot\dot{0}\dot{1})^{\frac{1}{5}}$, $(\cdot\dot{2}\dot{0})^{4}$.
11. To $(387{\cdot}908)^{\frac{1}{3}}$, $(\cdot 0187)^{\frac{2}{5}}$.

Ex. 69. Find the value—

1. Of $\dfrac{38067 \times \cdot 000507 \times 1{\cdot}3596}{\cdot 5498 \times 300 \times \cdot 0086735}$.
2. Of $\dfrac{281 \times 2{\cdot}71828 \times \cdot 09}{84000 \times \cdot 7301 \times \cdot 0073}$.
3. Of $\dfrac{\cdot 0084321 \times (\frac{2}{15})^{\frac{1}{3}}}{(8{\cdot}37)^{\frac{1}{2}}}$.
4. Of $\dfrac{(\cdot 005234)^{\frac{2}{7}} \times (\cdot 017)^{\frac{1}{3}}}{(24)^{\frac{1}{3}}}$.
5. Of $\left\{\dfrac{(15{\cdot}92)^{\frac{1}{7}} \times (\cdot 0182)^{\frac{1}{3}}}{\cdot 00526 \times (196)^{5}}\right\}^{\frac{1}{2}}$.
6. Of $\left\{\dfrac{15}{16}(\cdot 018)^{\frac{1}{7}}\right\}^{\frac{1}{11}}$.
7. Of $\left\{\dfrac{13659 \times (8{\cdot}256)^{10}}{(1{\cdot}86)^{\frac{5}{3}}}\right\}^{\frac{1}{100}}$.
8. Of $\dfrac{\frac{1}{6}(\frac{13}{14})^{\frac{1}{3}} \times \cdot 004(846)^{\frac{1}{7}}}{13(152)^{\frac{1}{2}} \div \cdot 34(\cdot 186)^{\frac{1}{9}}}$.
9. Of $\dfrac{(1{\cdot}025)^{13} - 1}{(1{\cdot}025)^{13} + 1}$.
10. Of $\dfrac{(1{\cdot}0975)^{11} - (1{\cdot}015)^{7}}{(24871{\cdot}53)^{\frac{1}{5}}}$.
11. Of $\dfrac{\cdot 2 \times \cdot 4 \times \cdot 8 \ldots \text{ to 12 terms}}{\cdot 3 \times \cdot 9 \times 2{\cdot}7 \ldots \text{ to 9 terms}}$.
12. Of $(£32\ 16s.\ 7\tfrac{3}{4}d.) \times (1{\cdot}015)^{50}$.

Ex. 70.

1. Given $\log 2 = \cdot 301030$, $\log 3 = \cdot 477121$, $\log 7 = \cdot 845098$; find the logs of 6, 15, 5·4, 17·5, ·875 and 6860.

2. Given $\log 18 = 1\cdot 255272$, $\log 25 = 1\cdot 397940$; find the logs of 2, 3, ·16, 450, ·075 and 3·75.

3. Find log 256 to the base $2\sqrt{2}$.

EXPONENTIAL EQUATIONS.

Ex. 71. Solve the equations—

1. $20^x = 100$.
2. $2^x = 769$.
3. $\left(\frac{1}{2}\right)^x = \frac{2}{3}$.
4. $\left(\frac{5}{4}\right)^x = 54\frac{1}{2}$.
5. $(2^3)^x \times (3^2)^x = 4\cdot 9$.
6. $\left(7^{\frac{1}{2}} \times 9^{-\frac{1}{3}}\right)^{2x} = 2^{\frac{1}{3}} \times 6^{\frac{1}{2}}$.
7. $\left(4\frac{1}{2}\right)^{(3\cdot 5)x} = 1000$.
8. $\left(\frac{3}{\dot{6}73}\right)^{-\frac{1}{x}} = 1\cdot 75$.
9. $a^{bx+d} = c$.
10. $a^{mx} . b^{nx} = c$,
11. $3^{2x} . 5^{3x-4} = 7^{x-1} . 11^{2-x}$.
12. $2^{3x} . 7^{4x-1} = 13^{5-x} . 17^{2x-1} . 19^x$.
13. $\left.\begin{array}{l} 14^x = 63y, \\ 17^x = 87y. \end{array}\right\}$
14. $\left.\begin{array}{l} 19^x = 53y, \\ 11^x = 31y. \end{array}\right\}$
15. $\left.\begin{array}{r} 2^x . 3^y = 560, \\ 5x = 7y. \end{array}\right\}$
16. $\left.\begin{array}{r} 3^{x+y} . 2^{-x} = 20, \\ 2x = 5y. \end{array}\right\}$
17. $\left.\begin{array}{l} 5^{3x} . 7^{2y} = 9156, \\ 3^x . 11^y = 3497. \end{array}\right\}$
18. $\left.\begin{array}{l} (a^x)^x . (b^y)^y = c, \\ nx = my. \end{array}\right\}$
19. $x^y = y^x$, and $x^3 = y^2$.
20. $(a^4 - 2a^2b^2 + b^4)^{x-1} = (a-b)^{2x} . (a+b)^{-2}$.
21. $2a^{4x} + a^{2x} = a^{6x}$.
22. $a^x - a^{-x} = 2c$.
23. $a^1 . a^3 . a^5$. &c. $a^{2x-1} = n$.

INTEREST AND ANNUITIES.

Ex. 72.

1. Find the amount of £660 at 5 per cent. per annum, compound interest, payable quarterly for 6 years.

2. What is the amount of £2639 16*s.* $3\frac{1}{2}$*d.* in 5 years, at 4 per cent. per annum, compound interest, payable monthly?

Ex. 72.

3. What sum will amount to £1000 in 10 years at 5 per cent. per annum, compound interest?

4. What sum will amount to £839 7*s.* 1$\frac{1}{4}$*d.* in 15 years at 3$\frac{1}{2}$ per cent. per annum, compound interest, payable quarterly?

5. At what rate per cent. will £300 amount to £500 in 4 years, compound interest?

6. At what rate per cent. will a given sum double itself in 6 years, compound interest?

7. In how many years will £600 amount to £6000 at 5 per cent., compound interest?

8. In what time will a given sum treble itself, at 3 per cent. per annum, compound interest, payable half-yearly?

9. In how many years will £2653 7*s.* 6*d.* invested at 3$\frac{1}{2}$ per cent. per annum, compound interest, payable quarterly, amount to £3327 18*s.* 1·104*d.*?

10. What is the discount on £100 due 3 years hence, at 4$\frac{1}{2}$ per cent. per annum, compound interest?

11. A person puts out £25 at 4 per cent. per annum, compound interest, and adds to his capital at the end of every year a sum equal to the third part of the interest for that year; find the amount at the end of 20 years.

12. A sum of £3500 is left for three children, A, B and C, in such a manner that at the end of 7, 9, and 12 years, when they respectively will come of age, they are to receive equal sums; find the present values of each share at 4 per cent. per annum, compound interest.

13. A debt of £500 accumulating at 4 per cent. per annum, compound interest, is discharged in n years by annual payments of £41 13*s.* 4*d.*; find the value of n.

14. A banker borrows money at 3$\frac{1}{2}$ per cent. per annum, and pays the interest at the end of the year; he lends it out at the rate of 5 per cent. per annum, but receives the interest quarterly, and by this means gains £200 a-year; how much does he borrow?

15. A person spends in the 1st year n times the interest of his property; in the 2nd year $2n$ times that of the remainder; in the 3rd year $3n$ times that at the end of the 2nd year; and so on: at the end of $2t$ years he has nothing left; show that in the tth year he spends as much as he has left at the end of that year.

Ex. 72.

16. What is the present worth of an annuity of £425 at 5 per cent. per annum, payable quarterly, for 12 years, at compound interest?

17. A Freehold Estate yielding £330 a-year is sold for £6000; required the rate of interest allowed to the purchaser.

18. If a lease of $65\frac{1}{2}$ years be purchased for £250, what rent ought to be received that the purchaser may make 7 per cent. per annum on his money?

19. A person purchases the reversion of an estate after 12 years for £1000; what rent ought he to receive that he may realise 6 per cent. per annum on his money?

20. The reversion of an estate in fee simple producing £85 a-year is made over for the discharge of a debt of £946 12*s*. $7\frac{2}{3}$*d*.; how soon ought the creditor to take possession, if he be allowed 5 per cent. per annum interest for his debt?

21. Find the present value of an annuity of £25, to commence in 8 years, and then to continue for 15 years, at $3\frac{1}{2}$ per cent. compound interest.

22. An annuity of £50 is to commence at the end of 12 years and to continue for 25 years; find the equivalent annuity to commence immediately and to continue 25 years, at $3\frac{1}{4}$ per cent. per annum in both cases.

23. An annuity of £50 which is to continue for 36 years is left equally between A and B; A receives the whole for the first 12 years, and B the whole for the remainder of the time: what is the present worth of the annuity, and the rate of compound interest per annum?

THEORY OF EQUATIONS.

Ex. 1. Form the equation whose roots are—

1. $+5, -3$.
2. $3, -2, 7$.
3. $0, -1, 2, -5$.
4. $\pm\frac{2}{3}, \pm\frac{4}{5}$.
5. $0, \pm 7, \frac{1}{2}(1 \pm \sqrt{-1})$.
6. $\pm 4\sqrt{3}, 5 \pm 2\sqrt{-1}$.
7. $\pm\sqrt{2}, \pm\sqrt{-3}$.
8. $1 \pm a^{\frac{3}{2}}, \pm\sqrt{-c}$.
9. What is the 4th term of the equation whose roots are $-2, -1, 1, 3, 4$?
10. Find the middle term of the equation whose roots are $5, 3, 1, -1, -2, -4$.
11. Determine which of the Nos. 7, 6, 5, 4, 2, are roots of the equation $x^4 - 19x^3 + 128x^2 - 356x + 336 = 0$.
12. Investigate the 1st, 2nd, &c. derived polynomials of $x^5 - 10x^4 + 29x^3 - 10x^2 - 62x + 60$; and $x^5 - px^3 - qx^2 + s$.

Ex. 2. Find the equation containing the other roots of—

1. $x^4 - 19x^3 + 132x^2 - 302x + 56 = 0$, one root being 4.
2. $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$, two rts. being 1, 5.
3. $x^5 - 2x^4 - 67x^3 + 200x^2 + 588x = 1440$, two rts. being 2, 6.
4. $x^6 + 33x^3 + 148x + 240 = x^5 + 21x^4 + 40x^2$, two rts. being $3, -1$.
5. $x^9 + x^8 - 9x^7 + 3x^6 - 8x^5 + 8x^4 - 3x^3 + 9x^2 - x = 1$, one rt. being 1.

Ex. 3. Find all the roots—

1. Of $x^3 - 11x^2 + 37x - 35 = 0$, one root being $3 + \sqrt{2}$.
2. Of $x^4 - 3x^2 - 42x - 40 = 0$, one rt. being $-\frac{1}{2}(3 + \sqrt{-31})$.
3. Of $x^5 - 10x^4 + 29x^3 - 10x^2 - 62x + 60 = 0$, two rts. being $3, \sqrt{2}$.
4. Of $x^3 - 7x^2 + 16x - 12 = 0$, two rts. being equal.
5. Of $x^3 - 5x^2 + 8x - 4 = 0$, two rts. being equal.
6. Of $x^3 - x^2 - 8x + 12 = 0$, two rts. being equal.
7. Of $x^4 - \frac{1}{2}x + \frac{3}{16} = 0$, two rts. being equal.
8. Of $x^4 + 13x^3 + 33x^2 + 31x + 10 = 0$, three rts. being equal.

Ex. 3. Find all the roots—

9. Of $x^4-14x^3+61x^2-84x+36=0$, having 2 pairs of equal rts.
10. Of $x^5-13x^4+67x^3-171x^2+216x-108=0$, the rts. being of the form a, a, a, b, b.
11. Of $x^3-9x^2+23x-15=0$, the rts. being in Ar. Prog.
12. Of $x^3-3x^2-6x+8=0$, the rts. being in Ar. Pr.
13. Of $x^4-10x^3+35x^2-50x+24=0$, the rts. being in Ar. Pr.
14. Of $x^4-2x^3-21x^2+22x+40=0$, the rts. being in Ar. Pr.
15. Of $x^3-13x^2+39x-27=0$, the rts. being in Geom. Pr.
16. Of $x^3-14x^2+56x-64=0$, the rts. being in Geom. Pr.
17. Of $x^3-26x^2+156x-216=0$, the rts. being in Geom. Pr.
18. Of $x^4+px^3+qx^2+rx+s=0$, the rts. being in Geom. Pr.
19. Of $x^4-15x^3+70x^2-120x+64=0$, the rts. being in Geom. Pr.
20. Of $6x^4-35x^3+62x^2-35x+6=0$, the rts. being of the form $a,\ \frac{1}{a},\ b,\ \frac{1}{b}$.

TRANSFORMATIONS.

Ex. 4. Transform into equations having integral coefficients—

1. $x^3+2x^2+\frac{1}{4}x+\frac{1}{9}=0$.
2. $x^3-\frac{7}{3}x^2+\frac{11}{36}x-\frac{25}{72}=0$.
3. $x^4-\frac{5}{6}x^3+\frac{5}{12}x^2-\frac{7}{150}x-\frac{13}{900}=0$.
4. $x^4-\frac{4}{3}x^2-\frac{3}{8}x+\frac{5}{72}=0$.

Form the equation whose roots are the roots—

5. Of $x^4+7x^2-4x+3=0$, each multiplied by 3.
6. Of $x^4+2x^3-7x-1=0$, each multiplied by 5.
7. Of $x^3-3x^2+4x+10=0$, each divided by 2.
8. Of $x^3+18x^2+99x+81=0$, each divided by -3.
9. Of $x^3-27x-36=0$, each diminished by 3.
10. Of $x^4-18x^3-32x^2+17x+9=0$, each diminished by 5.
11. Of $x^5+2x^4-15x^3-12x^2-76x=80$, each diminished by 2.
12. Of $x^4-27x^2-14x+120=0$, each diminished by 3.
13. Of $x^4-18x^3-32x^2+17x+9=0$, each increased by 2.
14. Of $x^5-7x^3+2x-8=0$, each increased by 1·2.
15. Of $2x^4-13x^2+10x-19=0$, each diminished by 1.
16. Of $19x^4-22x^3-35x^2-16x-2=0$, each diminished by 3.

Ex. 5. Transform the following, into equations wanting the 2d term—

1. $x^3-6x^2+7x-2=0$.
2. $x^3-6x^2+5=0$.
3. $x^3-6x^2+12x+19=0$.
4. $x^3+9x^2-4x+8=0$.
5. $x^3-6x^2+4x-7=0$.
6. $x^3-2x^2-5x+1=0$.
7. $x^3+4x^2+3x-7=0$.
8. $x^4+8x^3+x^2-x-10=0$.
9. $x^4-3x^3+5x-6=0$.
10. $x^4-12x^3+17x^2-9x+7=0$.
11. $3x^3+15x^2+25x-3=0$.
12. $x^5+3x^4-7x^2-2x+5=0$.

Transform the following, into equations wanting the 3rd term—

13. $x^3+5x^2+8x-1=0$.
14. $x^3-6x^2+9x-20=0$.
15. $x^3-4x^2+5x-2=0$.
16. $x^4-18x^3-60x^2+x-2=0$.
17. $3x^4-4x^3+2x^2+7x-9=0$.
18. $2x^5+5x^4+5x^3-x^2+1=0$.
19. In an equation of n dimensions, the 2nd and 3rd terms may be taken away by the same transformation, when the square of the sum of the roots : sum of their squares $=n:1$.

Symmetrical Functions.

Ex. 6. If S_r denote the sum of the rth powers of the roots of an equation; find the value—

1. Of S_6 in $x^3-x-1=0$.
2. Of S_6 and S_{-2} in $x^4+x^3-7x^2-x+6=0$.
3. Of S_5 and S_{-3} in $x^4-x^3-19x^2+49x-30=0$.

Ex. 7. Find the equation whose roots are—

1. The sums of every 2 roots of $x^3-40x+39=0$.
2. The squares of the roots of $x^3-2x^2+2x-4=0$.
3. The squares of the roots of $x^3-6x^2+8x-10=0$.
4. The cubes of the roots of $x^3-2x^2+1=0$.

Ex. 8. If a, b, c be the roots of $x^3-px^2+qx-r=0$; find the equation whose roots are—

1. $a+b, \quad a+c, \quad b+c$.
2. $ab, \quad ac, \quad bc$.
3. $a^2, \quad b^2, \quad c^2$.
4. $a^2+b^2, \quad a^2+c^2, \quad b^2+c^2$.
5. $\frac{1}{a^2}, \quad \frac{1}{b^2}, \quad \frac{1}{c^2}$.
6. $\frac{a}{b}+\frac{b}{a}, \quad \frac{a}{c}+\frac{c}{a}, \quad \frac{b}{c}+\frac{c}{b}$.
7. $a^3, \quad b^3, \quad c^3$.

Reciprocal or Recurring Equations.

Ex. 9. Solve the following equations—

1. $x^4-10x^3+26x^2-10x+1=0$.
2. $x^4+5x^3+2x^2+5x+1=0$.
3. $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0$.
4. $x^5-5x^4+9x^3+9x^2-5x+1=0$.
5. $x^5-8x^4-8x+1=0$.
6. $x^5-\frac{15}{2}x^4+\frac{37}{2}x^3-\frac{37}{2}x^2+\frac{15}{2}x-1=0$.
7. $x^5-6x^4-2x^3+2x^2+6x-1=0$.
8. $x^6+3x^5-5x^4-2x^3-5x^2+3x+1=0$.
9. $x^8+4ax^6+2x^4-4ax^2+1=0$.
10. $x^9+x^8-9x^7+3x^6-8x^5+8x^4-3x^3+9x^2-x-1=0$.
11. $x^{10}-3x^8+5x^6-5x^4+3x^2-1=0$.
12. $2x^4-5x^3+6x^2-5x+2=0$.
13. $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0$.
14. $x^4\pm1=0$; $x^6-1=0$; $x^9+1=0$; $x^{12}-a^{12}=0$.
15. Two roots of $x^7-23x^5+22x^4+55x^3-32x^2-33x+10=0$ are 2 and -5; find all the other roots.

Equal Roots.

Ex. 10. Solve the following equations, having *equal* roots—

1. $x^3-2x^2-15x+36=0$.
2. $x^4-9x^2+4x+12=0$.
3. $x^4-6x^3+12x^2-10x+3=0$.
4. $x^4-12x^3+50x^2-84x+49=0$.
5. $2x^4-12x^3+19x^2-6x+9=0$.
6. $4x^4-4x^3-7x^2+8x-2=0$.
7. $6x^4-25x^3+26x^2+4x-8=0$.
8. $x^5-11x^4+44x^3-76x^2+52x-12=0$.
9. $x^5-x^4+4x^3-4x^2+4x-4=0$.
10. $x^6-2x^5-4x^4+12x^3-3x^2-18x+18=0$.
11. $x^6+3x^5-6x^4-6x^3+9x^2+3x-4=0$.
12. $x^8-7x^7-2x^6+118x^5-259x^4-83x^3+612x^2-108x-432=0$.
13. If an equation has 2 equal roots, and the terms are multiplied by the terms of an Ar. Prog. in order; show that the result is $=0$.

Ex. 10.

14. If an equation has t equal roots, the equation formed by multiplying the terms by the terms of an Ar. Prog. in order, has $(t-1)$ of them.

15. If $x^5+qx^3-rx^2-t=0$ has two equal roots; show that one of them is a root of $x^2+\frac{2q^2}{5r}x+\frac{5t}{3r}-\frac{4q}{15}=0$.

Ex. 11.

1. If the roots of $x^3-px^2+qx-r=0$ be in Harm. Progression, show that the mean root is equal to r divided by $\frac{1}{3}q$. Apply this property to solve $x^3-23x^2+135x-225=0$.

Solve the equations—

2. $x^3-11x^2+36x-36=0$; whose roots are in Harm. Prog.
3. $x^3-\frac{11}{6}x^2+x-\frac{1}{6}=0$; whose roots are in Harm. Prog.
4. $8x^3-6x^2-3x+1=0$; whose roots are in Harm. Prog.
5. $24x^3-26x^2+9x-1=0$; whose roots are in Harm. Prog.
6. $x^3-5x^2+16x-12=0$, and $x^3-2x^2-15x+16=0$; which have one root common to both.

N.B. If a *be the root common to both,* x—a *is a common measure.*

7. $x^3-5x^2+13x-9=0$, and $x^3-2x^2+4x-3=0$; which have one root common to both. (See the last note.)
8. $x^4-3x^3+4x^2-6x+4=0$; the product of two roots is 2.
9. $x^4+x^3-62x^2-80x+1200=0$; the product of two rts. is 30.
10. $x^3-17x^2+94x-168=0$; two roots are as $3:2$.

Cubic Equations.

Ex. 12. Solve by Cardan's or the Trigonometrical Method—

1. $x^3-9x-14=0$.
2. $x^3-9x+28=0$.
3. $x^3+6x-2=0$.
4. $x^3-3x-18=0$.
5. $x^3-9x^2+25x-25=0$.
6. $x^3+3x^2+9x-13=0$.
7. $x^3-6x^2+13x-10=0$.
8. $x^3-10x^2+31x-42=0$.
9. $x^3+6x^2-32=0$.
10. $x^3-3x^2-9x+20=0$.
11. $x^3-8x^2-6x+9=0$.
12. $x^3-6x^2+5x+12=0$.
13. $x^3-8x-1=0$.
14. $x^3-49x+120=0$.
15. $x^3+6x^2+27x-26=0$; the real root to 6 decimals.
16. $x^3-3x^2+5x-43=0$; the real root to 6 decimals.

Ex. 12. Solve by Cardan's or the Trigonometrical Method—

17. $x^3-13x^2+49x-45=0$; the real root to 6 decimals.
18. $x^3-9x^2+6x-2=0$; the real root to 4 decimals.
19. $x^{\frac{2}{3}}+\cdot24x^{\frac{1}{3}}-\cdot245=0$.
20. $x^{\frac{2}{n}}-36x^{\frac{1}{n}}-91=0$.

Biquadratic Equations.

Ex. 13. Solve by Des Cartes', Euler's or Ferrari's Method—

1. $x^4-3x^2-42x-40=0$.
2. $x^4-25x^2-60x-36=0$.
3. $x^4-12x-5=0$.
4. $x^4-4x^3-8x+32=0$.
5. $x^4-6x^3+8x^2+6x-9=0$.
6. $x^4+8x^3+9x^2-8x-10=0$.
7. $x^4-8x^3-12x^2+60x+63=0$.
8. $x^4+2x^3-7x^2-8x+12=0$.
9. $x^4-20x^3+148x^2-464x+480=0$.
10. $9x^4-6x^3-439x^2+294x-98=0$.

Limits of Roots.

Ex. 14. Determine a number greater than the greatest positive root—

1. Of $x^3-4x^2-4x+20=0$; by Newton's Method.
2. Of $x^3-5x^2+7x-1=0$; by Newton's Method.
3. Of $x^3-6x^2+18x-22=0$; by Newton's Method.
4. Of $x^4-2x^3-3x^2-15x-3=0$; by Newton's Method.
5. Of $x^4-5x^3+11x-20=0$; by Newton's Method.
6. Of $x^4+14x^2-12x-35=0$.
7. Of $x^5+x^4+x^2-25x-36=0$.
8. Of $x^5+7x^4-12x^3-49x^2+52x-13=0$.
9. Of $4x^5-8x^4+23x^3+105x^2-80x+3=0$.
10. Apply the consideration of signs to determine the number of positive and negative roots of $x^5-2x^4-4x+8=0$.

Rational Roots.

Ex. 15. Find, by the method of divisors, the roots—

1. Of $x^3-9x^2+22x-24=0$.
2. Of $x^3-2x^2-4x+8=0$.
3. Of $x^3+3x^2-8x+10=0$.
4. Of $x^3-5x^2-18x+72=0$.
5. Of $x^4-4x^3+8x^2-16=0$.
6. Of $2x^3-3x^2+2x-3=0$.
7. Of $x^4-x^3-13x^2+16x-48=0$.
8. Of $3x^3-2x^2-6x+4=0$.
9. Of $8x^3-26x^2+11x+10=0$.
10. Of $x^4-9x^3+\frac{45}{4}x^2+\frac{27}{2}x-\frac{81}{4}=0$.
11. Of $6x^4-25x^3+26x^2+4x-8=0$.
12. Of $6x^4+53x^3-95x^2-25x+42=0$.

Sturm's Theorem.

Ex. 16. Determine the number and position of the real roots—

1. Of $x^3+2x^2-3x+2=0$.
2. Of $x^3-3x^2-4x+11=0$.
3. Of $x^3-2x-5=0$.
4. Of $x^3-15x-22=0$.
5. Of $x^3-27x-36=0$.
6. Of $x^3-12x-15=0$.
7. Of $x^3+2x^2-51x+110=0$.
8. Of $x^3+10x^2+5x-2600=0$.
9. Of $x^4-4x^3-3x+23=0$.
10. Of $x^4-33x^2-100x-84=0$.
11. Of $x^4-18x^3+71x^2+100x-70=0$.
12. Of $x^4-12x^3+13x^2+24x-30=0$.
13. Of $x^4-6x^3+46x-105=0$.
14. Of $x^4-x^3-4x^2+4x+1=0$.
15. Of $x^5-3x^4-24x^3+95x^2-46x-101=0$.
16. Of $x^5+3x^4+2x^3-3x^2-2x-2=0$.
17. Of $x^5-x^4-15x^3+38x^2-26x+6=0$.
18. Of $x^5-10x^3+6x+1=0$.
19. Of $x^7-2x^5-3x^3+4x^2-5x+6=0$.
20. Of $x^6+24x^5+125x^4-376x^3-1726x^2+5592x-4080=0$.

Approximation.

Ex. 17. Determine a real root—

1. Of $x^3-2x-5=0$; by Newton's Method.
2. Of $x^3-5x-3=0$; by Newton's Method.
3. Of $x^3-7x-1=0$; by Newton's Method.

Ex. 17. Determine a real root—

4. Of $x^3-7x+7=0$; by Newton's Method.
5. Of $2x^3-3x-6=0$; by Newton's Method.
6. Of $6x^3-141x+263=0$; by Double Position.
7. Of $2x^4-13x^2+10x-19=0$; by Double Position.
8. Of $x^4-12x+7=0$; by Double Position.
9. Of $x^5-2x^4-13x^3+39x^2-20x+4=0$; by Double Position.
10. Of $5x^6-13x^3-1100=0$; by Double Position.
11. Of $(x^2+3x+1)^{\frac{1}{3}}+(2x^2+1)^{\frac{1}{3}}=2{\cdot}617$; by Double Position.
12. Of $x^x+5x=1000$; by Double Position.
13. Of $2x^4+5x^3+4x^2+3x=8002$; by Horner's Method.
14. Of $x^5+4x^4-2x^3+10x^2-2x=962$; by Horner's Method.
15. Of $x^5+6x^4-10x^3-112x^2-207x=110$; by Horner's Method.
16. Of $x^5+12x^4+59x^3+150x^2+201x=207$; by Horner's Method.
17. Approximate, by the Method of Continued Fractions, to the greatest root of $x^3-3x^2-2x+1=0$, determining the first four converging fractions.
18. Find the limiting equation of
$$(x-a)(x-b)(x-c)-m^2(x-a)-n^2(x-b)-p^2(x-c)-2m^2n^2p^2=0;$$
and thence show that all its roots are real.

GEOMETRY.

Miscellaneous Theorems and Problems.

1. The straight line which bisects the vertical angle of an isosceles triangle, bisects the base, and is perpendicular to it.

2. The straight line, which joins the vertex of an isosceles triangle and the middle of the base, bisects the vertical angle, and is perpendicular to the base.

3. If the line bisecting the vertical angle of a triangle be perpendicular to the base, the triangle is isosceles.

4. If the line bisecting the vertical angle of a triangle, also bisect the base, the triangle is isosceles.

5. The lines bisecting an internal, and the adjacent external angle of a triangle, are at right angles to each other.

6. From two given points, to draw two straight lines to meet in a given straight line, and to make equal angles with it.

7. Of any two straight lines, that may be drawn from two given points without a given line, to meet in that line, the sum is the least, when they make equal angles with the line.

8. On a given straight line, to describe a square, of which it shall be the diagonal.

9. To trisect a given finite straight line.

10. The difference between any two sides of a triangle is less than the third side.

11. If any two sides of a triangle be bisected, and from the middle points, perpendiculars be drawn to the sides, the straight lines joining the point of intersection with the three angular points of the triangle are equal to one another.

12. If any two angles of a triangle be bisected by straight lines, which meet in a point, the three perpendiculars drawn from this point to the sides of the triangle are equal to one another.

13. The three perpendiculars drawn from the middle points of the sides of a triangle, intersect in one point.

14. The three straight lines bisecting the angles of a triangle, intersect in one point.

15. The three straight lines joining the angular points of a triangle with the middle points of the opposite sides, intersect in one point.

16. In a right-angled triangle, the line drawn from the right angle to the middle point of the hypothenuse is equal to half the hypothenuse.

17. In a right-angled triangle, the angle contained by the line bisecting the right angle, and the line drawn perpendicular to the hypothenuse, is equal to half the difference of the two acute angles of the triangle.

18. If each of the equal angles of an isosceles triangle be equal to one-fourth the vertical angle, and from one of them a perpendicular be drawn to the base, meeting the opposite side produced, then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

19. The quadrilateral figure, whose diagonals mutually bisect each other, is a parallelogram.

20. The parallelogram, whose diagonals are equal, is rectangular.

21. The parallelogram, whose diagonals intersect at right angles, is equilateral.

22. Through a given point to draw a straight line which shall make equal angles with two given straight lines.

23. The area of a rhombus equals half the rectangle contained by the diagonals.

24. From a given point between two given straight lines, to draw a straight line, which shall be terminated by the given straight lines, and bisected by the given point.

25. If in the sides of a square, four points be taken, at equal distances from the four angular points taken in order; the figure contained by the straight lines which join them shall also be a square.

26. If the sides of any hexagon be produced to meet, the angles formed by these lines are together equal to four right angles.

27. If the sides of any pentagon be produced to meet, the angles formed by these lines are together equal to two right angles.

28. If the sides of any polygon being produced meet externally the angles formed by these lines are together equal to twice as many right angles wanting eight as the polygon has sides.

29. If from any angle of a triangle, a straight line be drawn to the middle point of the opposite side, the sum of the squares of the sides containing this angle shall be equal to twice the square of the bisecting line, together with twice the square of half the bisected side.

30. In a parallelogram, the sum of the squares of the sides is equal to the sum of the squares of the diagonals.

31. If from the angles of a triangle, lines be drawn bisecting the opposite sides, four times the squares of these lines is equal to three times the squares of the sides of the triangle.

32. In any triangle ABC, if BP, CQ be drawn perpendicular to CA, BA, produced if necessary, then shall $BC^2 = AB.BQ + AC.CP$.

33. If ABC be an isosceles triangle ($B = C$), and if CD be drawn perpendicular to AB, then shall $CD^2 = BD^2 + 2BD.DA$.

34. If AD be drawn to any point D in the base of the triangle ABC; show that $AB^2.CD + AC^2.BD = AD^2.BC + BC.BD.CD$.

35. If ABC be a triangle, with the angles at B, C, each double of the angle at A; prove that $AB^2 = BC^2 + AB.BC$.

36. If perpendiculars AP, BQ, CR be drawn from the angular points of a triangle ABC upon the sides, show that they will bisect the angles of the triangle PQR.

37. Bisect a triangle by a line drawn from a given point in one of its sides.

38. Bisect a triangle by a line drawn parallel to one side.

39. If the areas of a triangle and of a square be equal, the perimeter of the triangle will be the greater.

40. If from the angle A of any parallelogram, any line be drawn cutting the diagonal in P, and the sides BC, CD, produced if necessary in Q, R; show that $AP^2 = PQ.PR$.

41. If two circles touch, either externally or internally, and from the point of contact two chords be drawn, meeting the circumferences, the chords of the intercepted arcs will be parallel.

42. If two chords intersect in a circle, the difference of their squares is equal to the difference of the squares of the difference of the segments.

43. If two chords be drawn from any point of a circle, and upon these chords, as diameters, two other circles be described, the three points of intersection of these three circles will be in the same straight line.

44. A common tangent is drawn to two circles, which touch externally; if a circle be described on that part of it which lies between the points of contact, as diameter, this circle will pass through the common point of contact of the two circles, and be touched by the line which joins their centres.

45. If from any point without a circle, two straight lines be

drawn, making equal angles with the line through the centre, they will cut off equal segments from the circle.

46. DF is a straight line touching a circle, and terminated by AD, BF, tangents at the extremities of any diameter AB; show that the angle subtended by DF at the centre is a right angle.

47. If the chord of a quadrant be made the diameter of a semicircle, and from its extremities two straight lines be drawn to any point in the arc of the semicircle, the segment of the greater line between the two arcs is equal to the less line.

48. Show that there can be six equal circles placed around a given circle of the same diameter, so as to touch each other and the given circle.

49. If on two lines containing an angle, segments of circles be described, containing angles equal to it, the straight lines being produced, shall touch the circular arcs described on them exterior to the angle.

50. Of all lines which touch the interior and are bounded by the exterior of two circles which touch internally, the greatest is that which is parallel to the common tangent.

51. If two circles cut each other, the line joining their centres shall bisect their common chord.

52. If two circles cut each other, and from either point of intersection diameters be drawn, the lines joining the extremities of these diameters will pass through the other point of intersection.

53. If a common tangent be drawn to any number of circles which touch each other internally, and from any point of this tangent as a centre, a circle be described, cutting the other circles; and if from this centre, lines be drawn through the intersections of the circles, the segments of the lines within each circle shall be equal.

54. To draw two straight lines from two given points to meet in a line, given in position, and which shall contain a right angle.

55. To find a point, such that the tangents drawn from it, to touch two given circles that touch one another externally, may contain a given angle.

56. If a straight line touch the interior of two concentric circles, and be terminated by the exterior one, it will be bisected by the point of contact.

57. If from the extremities of any diameter of a given circle, perpendiculars be drawn to any chord of the circle, or the chord produced, that is not parallel to the diameter, the less perpendicular shall be equal to the segment of the greater contained between the circumference and the chord.

58. If a tangent to a circle be parallel to a chord, the point of contact is the middle point of the arc cut off by the chord.

59. Any two parallel lines meeting a circle cut off equal arcs.

60. If two chords of a circle intersect at right angles, the sum of the squares of the four segments equals the square of the diameter.

61. If two chords of a circle intersect at right angles, and thus cut off four arcs, the sum of any two opposite arcs equals the sum of the other two.

62. AB is the diameter of a circle, MN a chord parallel to AB; in AB take any point P, and join PM, PN, then shall the sum of the squares of PM and PN equal the sum of the squares of AP and BP.

63. If a circle be described on the radius of another circle, as its diameter, any straight line drawn from the common point of contact, and terminated by the outer circumference, is bisected by the inner one.

64. In every triangle inscribed in a circle, the intersections of the sides produced and tangents at the angles opposite, are in a straight line.

65. AB is any chord of a circle; AC, BC are drawn to any point C in the circumference, and cut the diameter perpendicular to AB in D, E; if O be the centre, show that $OD.OE=OA^2$.

66. If ACDB be a semicircle, whose diameter is AB, and AD, BC be any two chords intersecting in P; show that

$$AB^2=AD.AP+BC.BP.$$

67. ABC is a triangle whose acute vertex is A; show that the square of BC is less than the squares of AB, AC, by twice the square of the line drawn from A to touch the circle on BC as diameter.

68. Two circles touch one another externally in C; if any point D be taken without them, such that the radii AC, BC subtend equal angles at D, and DE, DF be tangents to the circles, then $DE.DF=DC^2$.

69. To draw a straight line which shall touch two given circles.

70. If from any point without a circle two lines be drawn touching it, the angle contained by these lines is double the angle contained by the chord joining the points of contact, and the diameter drawn from one of these points.

71. To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square of a given line, which is less than the line to be divided.

72. To draw a straight line, which shall touch a given circle, and make with a given straight line an angle equal to a given angle.

73. To describe a circle which shall pass through a given point, and touch a given straight line in a given point.

74. To describe a circle which shall pass through a given point, have a given radius, and touch a given straight line.

75. To describe a circle which shall pass through a given point, and touch a given circle in a given point.

76. To describe a circle which shall touch a given straight line in a given point, and also touch a given circle.

77. To describe a circle which shall touch a given circle in a given point, and also touch a given straight line.

78. To describe a circle which shall have a given radius, and its centre in a given straight line, and shall also touch another straight line inclined at a given angle to the former.

79. If an equilateral triangle be inscribed in a circle, and chords be drawn from its angles to any one point in the circumference, that chord which falls within the triangle shall be equal to the sum of the other two.

80. From the obtuse angle of a triangle, to draw a straight line to the base, such that it may be a mean proportional between the segments of the base.

81. To divide a right angle into five equal parts.

82. To inscribe a circle in a given rhombus.

83. To inscribe a circle in a given segment of a circle.

84. The area of an equilateral triangle inscribed in a circle is equal to one-fourth of that described about it.

85. The area of the inscribed square is equal to one-half of that of the circumscribed.

86. The area of the inscribed regular hexagon is equal to three-fourths of that of the circumscribed.

87. If ABC be an isosceles triangle, and DE be drawn parallel to the base BC, then shall $BE^2 = BC.DE + CE^2$.

88. ABC is an equilateral triangle, E any point in AC; in BC produced take $CD = CA$, $CF = CE$, and let AF, DE intersect in H; prove that $HC(AC + EC) = AC.EC$.

89. If a triangle ABC have a right angle C, and AD be drawn bisecting the angle A, and meeting BC in D; then shall $AC^2 : AD^2 = BC : 2BD$.

90. Prove also, that in the same triangle $2AC^2 : AC^2 - CD^2 = BC : CD$.

91. APB is the quadrant of a circle, SPT a line touching it at P; C is the centre, PM perpendicular to CA; prove that

triangle SCT : triangle ACB = triangle ACB : triangle CMP.

92. In any triangle ABC, if AE be drawn from the angle A in any direction, and BE, CF be drawn perpendiculars to it from B and C, and EG, FG be drawn to the point of bisection G of BC; then shall GE equal GF.

93. If perpendiculars be drawn from the extremities of the base of a triangle to a straight line, which bisects the angle opposite to the base, the area of the triangle is equal to the rectangle contained by either of the perpendiculars, and the segment of the bisecting line that is between the angle and the other perpendicular.

94. If from any point within an equilateral triangle, perpendiculars be drawn to the sides, the sum of these perpendiculars is equal to the perpendicular drawn from either of the angles to the opposite side.

95. ABC is a triangle inscribed in a circle, AD, AE lines drawn to the base and parallel to the tangents at B, C respectively; show that $AD = AE$, and $BD : CE = AB^2 : AC^2$.

96. In any triangle ABC, right-angled at C, if AD be drawn bisecting the angle A; show that $BC : AC = AB - AC : CD$.

97. ABC is a triangle inscribed in a circle; AD, AE are lines drawn to BC or BC produced, parallel to the tangents at C, B respectively; show that $BE : CD = AB^2 : AC^2$.

98. ABCD is a trapezoid, AD parallel to BC; show that

$$AC^2 \sim BD^2 : AB^2 \sim CD^2 = BC + AD : BC \sim AD.$$

99. AB, AC are the sides of a regular pentagon and decagon inscribed in a circle whose centre is O; if OP be drawn to AB bisecting the angle AOC; show that the triangles ABC, APC, as also the triangles AOB, BOP are similar, and that $AB^2 - AC^2 = AO^2$.

100. The diagonals AC, BD of an inscribed quadrilateral meet in E; show that $AB.BC : AD.DC = BE : ED$.

101. AB, the line joining the centres of two circles whose radii are R, r, is divided in C, so that $AB : R + r = R - r : AC - BC$; show that the tangents drawn to the circles from any point in CD, perpendicular to AB, are equal.

102. To find two straight lines which shall be arithmetic means between two given straight lines.

103. To divide a given straight line AB by two points of division C and D, so that AC, AD, AB may be in harmonical proportion.

104. If through any point D of a straight line AD which bisects a given angle BAC a straight line GFDE be drawn meeting AG

(which is drawn perpendicular to AD), AB and AC respectively in G, F, E; show that ED, EF, EG are in harmonical proportion.

105. If the four sides of a quadrilateral figure be bisected, the lines joining the points of bisection shall form a parallelogram, whose area equals half the area of the quadrilateral.

106. If two diagonals of a regular pentagon intersect; 1st, the greater segment is equal to a side of the pentagon; 2nd, the two diagonals cut each other in extreme and mean ratio.

107. Within an isosceles triangle to find a point, such that its distance from one of the equal angles may equal twice its distance from the vertical angle.

108. If two circles touch each other, and any two parallel diameters be drawn, the straight line joining their extremities towards the same or opposite parts, according as the circles touch internally or externally, shall pass through the point of contact.

109. If two circles touch each other externally, and also a given straight line, the part of the line between the points of contact is a mean proportional between the diameters.

110. If two circles touch each other, either internally or externally, any two straight lines drawn from the point of contact will be cut proportionally by the circumferences.

111. If from one extremity of a chord a tangent be drawn to a circle, equal to the chord, and a line be drawn joining the further extremities of the chord and the tangent, the arc intercepted between that line and the tangent shall be equal to half the arc subtended by the chord.

112. If a straight line, which touches two circles, cut another straight line, which joins their centres, the segments of the latter will be proportional to the diameters.

113. If from the extremities of any chord of a circle, perpendiculars be drawn to the chord, the points where they meet any diameter shall be equally distant from the centre.

114. If a circle be inscribed in a triangle, and another circle be described touching the base and the other two sides produced; 1st, the points where the circles touch the base shall be equally distant from its extremities; 2nd, the distance between the points where they touch either one of the sides shall be equal to the base.

115. To describe a circle which shall pass through two given points and touch a given straight line.

116. From a given point in the side of a triangle, to draw a straight line, which shall bisect the triangle.

117. From a given angle of a trapezium, to draw a straight line, bisecting the trapezium.

118. From a given point in the side of a triangle, to draw straight lines, which shall divide it into any number of equal parts.

119. To transform any rectilineal figure into a triangle, of equal area, whose vertex shall be in one of the angles of the figure, and its base in one of its sides.

120. To transform any given triangle into an isosceles one of equal area.

121. To transform any given isosceles triangle into an equilateral one of equal area.

122. To divide a given straight line into two segments, such that the rectangle contained by them shall be a maximum.

123. Through a given point within a circle, to draw the least possible chord.

124. On a given base, to describe a triangle, having a given vertical angle, and whose area shall be a maximum.

125. On a given base, to describe a triangle, another of whose sides is given, so that the area may be a maximum.

126. To divide a circle into any number of parts, which shall be equal both in area and in perimeter.

127. To divide a circle into any number of equal concentric annuli.

128. Describe a square, having given the difference between the diagonal and a side.

129. Given one side of a right-angled triangle, and the difference between the hypothenuse and the other side, to construct it.

130. Given the perpendicular from the right angle on the hypothenuse of a right-angled triangle, and the difference of the segments of the hypothenuse, to construct it.

131. Given the hypothenuse, and the sum of the two sides of a right-angled triangle, to construct it.

132. Given the segments of the hypothenuse of a right-angled triangle, made by a perpendicular from the right angle, to construct it.

133. Given the hypothenuse, and the difference of the sides of a right-angled triangle, to construct it.

134. Given the sides of a right-angled triangle in continued proportion, and the length of the hypothenuse, to construct it.

135. Given the base of a triangle, one of the angles at the base, and the difference of the two sides, to construct it.

136. Given the base, the difference of the two angles at the base, and the difference of the two sides of a triangle, to construct it.

137. Given the vertical angle of a triangle, and the segments into which the perpendicular from the vertex divides the base, to construct it.

138. Given the base, the vertical angle, and the sum of the sides of a triangle, to construct it.

139. Given the base, the ratio of the sides, and the vertical angle of a triangle, to construct it.

140. Given the vertical angle of a triangle, the sum of its sides, and the difference of the segments into which a perpendicular from the vertex divides the base, to construct it.

MENSURATION.

AREAS OF PLANE FIGURES.

General Formulæ.

1. In a parallelogram, if b, c be two sides including the angle A, d the perpendicular on b;

$$\text{area} = bc \sin A = bd.$$

2. In a triangle, if a, b, c be the sides opposite to the angles A, B, C; d the perpendicular from A on a, p the semi-perimeter;

$$\text{area} = \tfrac{1}{2}ad = \tfrac{1}{2}bc \sin A = \sqrt{p(p-a)(p-b)(p-c)}.$$

3. In a trapezoid, if a, b be the two parallel sides, d the perpendicular distance between them;

$$\text{area} = \frac{d}{2}(a+b).$$

4. In a regular polygon of n sides, if $2a$ be the length of each side;

$$\text{area} = na^2 \cot \frac{180^\circ}{n}.$$

5. In a circle, if r be the radius, and $\pi = 3{\cdot}1416$;

$$\text{circumference} = 2\pi r; \qquad \text{area} = \pi r^2.$$

6. In a circular ring, if a and b be the external and internal radii;

$$\text{area} = \pi(a^2 - b^2).$$

7. In the sector of a circle, if n be the number of degrees at the centre,

length of arc : circumference of circle : : n : 360,

area of sector : area of circle : : n : 360,

area of sector $= \frac{1}{2}(\text{arc} \times \text{radius})$.

8. In a parabola, if a be the height, and b the base;

$$\text{area} = \tfrac{2}{3}ab.$$

9. In an ellipse, if a and b be the semi-axes;

$$\text{area} = \pi ab.$$

N.B. 100 links = 1 chain, and 10 square chains = 1 acre.

PARALLELOGRAM.

Ex. 1. Find the area of—

1. A square, whose side is 15 chains 40 links.
2. A rectangular field, whose sides are $50\frac{3}{4}$ and 123 yards.

Ex. 1. Find the area of—

3. A rhombus, whose side is 5 ft. 7 in., and height 4 ft.

4. A rhombus, whose side is 17 yd., and angle $49° 14' 15''$.

5. A rhomboid, the base being 23 ft. 8 in., and height 16 ft. 9 in.

6. A rhomboid, whose sides are 4 and $5\frac{1}{2}$ chains, angle $16° 43'$.

7. The side of a square court-yard is 85 ft. 3 in.; what will it cost paving at 2*s*. 9*d*. per square yard?

8. What is the side of a square garden that cost £33 16*s*. $10\frac{1}{2}d$. trenching, at $2\frac{1}{2}d$. per square yard?

9. Find the side of a square field of 10 acres.

10. If the length of a rectangular field, whose area is 29 acres 22 poles, be 18 chains 50 links; what is the breadth?

11. The side of a rhombus is 20, and its longer diagonal 34·64; find the area, and the other diagonal.

12. The two diagonals of a rhombus are 50·4 and 37·8; find the side and the area.

13. A grass-plot, in the form of a rhombus, cost 5 guineas making at 7*d*. per square yard; if the side be 45 feet, what is the angle?

14. How much paper $\frac{3}{4}$ yard wide will be required for a room, that is 22 feet long, by 14 feet wide, and 9 feet high; if there be 3 windows and 2 doors, each 6 feet by 3 feet?

TRIANGLE.

Ex. 2. Find the area of a triangle—

1. The base being 8 ft. and height 7 ft. 9 in.

2. Two sides being 24, 17·6 yd., and the included angle 30°.

3. Isosceles, the vertical angle 120°, and height 6 ft. 3 in.

4. Equilateral, each side being 18 ft.

5. The sides being 400, 348 and 312 yd.

6. The sides being 3615, 2709 and 2874 links.

7. Equilateral, the perimeter being 125 ft.

8. Two angles being $57° 50'$, $38° 40'$, and side opposite the latter, 72.

9. Find the side of an equilateral triangle whose area is 5 acres.

10. Find the side of an equilateral triangle whose area cost as much paving at 9*d*. per foot, as palisading the three sides did at 15*s*. a yard.

Ex. 2.

11. A triangular field 738 links long, and 583 links in the perpendicular, produces an income of £12 a year. At how much an acre is it let?

12. The paving of a triangular court-yard came to £100, at 15*d.* per square foot; if one of the sides be 24 yards long, find the length of each of the other two equal sides.

13. Two sides of a triangle, whose area is 6 acres, being each equal to 275 yards; find the angle between them.

Trapezoid and Trapezium.

Ex. 3.

1. How many square feet are there in a plank, whose length is 10 ft. 5 in., and the breadths of the two ends $2\frac{1}{2}$ ft. and $1\frac{3}{4}$ ft.?

2. Find the area of a trapezoid, whose parallel sides are 72 and $38\frac{2}{3}$ feet; the other sides being 20 and $26\frac{2}{3}$ feet.

3. If one of the parallel sides of a trapezoid be 137, and the angles at its extremities $46° 15'$ and $54° 12'$, also the altitude 36; find the area.

4. If the area of a trapezoid be 542 square feet, one of the parallel sides 64 feet, and the angles at its extremities $72° 16'$ and $58° 42'$; find the other parallel side and the altitude.

5. The parallel sides of a trapezoid are 37 and 19, and the angles made by the other sides with the side (37) are 68° and $41° 20'$; find the area.

6. The breadth of the bottom of a ditch is to be 16 feet, the depth 9 feet, and the inclinations of the sides to the top 65° and 74°; what must be the breadth of the excavation at the top?

7. How many square yards of paving are there in a quadrangular court, whose diagonal is 54 feet; and the perpendiculars on it from the opposite corners 25 and $17\frac{3}{4}$ feet respectively?

8. The sides of a quadrilateral field taken in order are 1208, 856, 974, 1424 links; the angle included between the first two is $78° 40'$, and between the last two $63° 55'$; find the area.

9. The sides of a trapezium are 690, 467, 359, 428 yd.; the angle between the first and second sides is $57° 30'$, and between the third and fourth $122° 30'$; find the area.

10. In the trapezium ABCD, if AB = 345, BC = 156, CD = 323, DA = 192, and the diagonal AC = 438; find the area.

11. Find the area, in acres, of a quadrilateral field ABCD, if AD = 220, BC = 265, and the diagonal AC = 378 yards; and if

Ex. 3.

perpendiculars from D and B meet the diagonal in E, F so that AE = 100, CF = 70 yards.

12. The sides of a trapezium are 335, 426, 387, 321 yd., and the angle contained by the first two is a right angle; find the area.

REGULAR POLYGON.

Ex. 4. Find the areas of the following polygons—

1. A pentagon, of which each side is 15 feet.
2. A hexagon, of which each side is 30 feet.
3. A heptagon, of which each side is 45 feet.
4. An octagon, of which each side is 15 feet.

Find the side of the regular polygon, in—

5. A heptagon, of which the area is one acre.
6. An octagon, of which the area is one rood.
7. A decagon, of which the area is four perches.
8. A dodecagon, of which the area is 1000 sq. yards.

Find the radii of the inscribed and circumscribed circles—

9. Of a pentagon, each side of which is 3.
10. Of a heptagon, each side of which is $25\frac{1}{4}$.
11. Of an undecagon, each side of which is 20.

12. If the radius of a circle be 50, find the sides of a regular inscribed and circumscribed pentagon, octagon, and dodecagon.

13. Find the area of a regular nonagon inscribed in a circle, of which the radius is 12.

14. A regular polygon of 25 sides is described about a circle of radius 10; find its area.

CIRCLE.

Ex. 5.

1. The diameter of a circle is 5 feet; what is its circumference?
2. The circumference of a circle is 10 chains; what is its radius?
3. What is the area of a circle, whose diameter is 12 feet?
4. Find the area of a circle, whose circumference is one mile.
5. The area of a circle being one acre, what is its radius?

6. How much will the turfing of a round plot cost at 4*d.* per square yard, if it be 130 feet round?

Ex. 5.

7. The paving of a semicircular alcove with marble at 2*s*. 6*d*. a foot came to £10; what was the length of the semicircular arc?

Find the area of a circular ring—

8. The internal and external diameters being 5 and 15 ft.

9. The radii of the bounding circumferences being 16 and 24 ft.

10. The external circumference being 55 inches, the width $\frac{1}{2}$ inch.

11. A curb 15 inches broad is put to a well 7 feet in diameter, and costs 17*s*. 6*d*. At how much per square foot is that?

12. Three men bought a grindstone one yard in diameter; the shares being equal, what part of the diameter may each man grind down?

Find the length of an arc of a circle—

13. The radius being 9, and the angle at the centre $29\frac{1}{2}°$.

14. The chord being 45, and chord of half the arc 25·5.

15. The chord being 24, and height 9.

16. Find the length of the minute-hand of a dial, the extremity of which moves over an arc of 5 inches in $3\frac{1}{4}$ minutes.

Ex. 6. Find the area of the Sector of a circle, when—

1. The radius is 8 feet, and the arc contains 159°.
2. The radius is 30 yards, and the arc 63 feet.
3. The arc contains 39° 30′, and is 10 feet long.
4. The radius is 15, and the chord 12.
5. The chord is 20, and the angle at the centre 70° 18′.
6. The chord is 18, the perpendicular on it from the centre 10.
7. The chord of half the arc is 17, and the height 8.
8. If the area of a sector be 15 square feet, and the length of the arc 6 feet; find the radius.
9. If the area of a sector be 20 square feet, and the radius 5 feet; find the number of degrees at the centre.
10. A field in the form of an equilateral triangle contains half an acre; what must be the length of a tether fixed at one of its angles, and to a horse's nose, to enable him to graze exactly half of it?

Ex. 7. Find the area of the Segment of a circle, when—

1. The diameter is 7 feet, and the height 30 inches.
2. The radius is 12, and the chord 16.

Ex. 7. Find the area of the SEGMENT of a circle, when—

3. The radius is 20, and the angle at the centre 36°.

4. The chord is 22, and the height 22.

5. The chord is 30, and the height 9.

6. The chord is 150, and the arc contains 112° 30′.

7. The chord is 40, and the chord of half the arc 25.

8. The arc is a quadrant, and radius 24.

9. The arc contains 120°, and is 500 feet long.

10. If the centre of a circle whose diameter is 16, be in the circumference of a circle whose diameter is 30; find the area of the figure common to both circles.

11. If two circles be described on the bounding radii of a quadrant of a circle whose diameter is 4; find the area common to the two circles, and the area intercepted between the arc of the quadrant and the circumferences of the two circles.

12. Find the area of a parabola, whose height is 7 and base 12.

13. What is the area of an ellipse, whose axes are 34 and 30 feet?

14. The axes of an ellipse are 25 and 35; find the area of an elliptic segment cut off parallel to the shorter axis, the height being 10.

SURFACES AND CONTENTS OF SOLIDS.

General Formulæ.

1. In a parallelopiped, if a, b, c be the three dimensions,
surface $=2(ab+ac+bc)$; volume $=abc$.

2. In a prism or cylinder, if p be the perimeter of either end, a^2 its area, h the height,
Lateral or convex surface $=ph$; volume $=a^2h$.

3. In a pyramid or cone, if p be the perimeter of the base, a^2 its area; h the perpendicular height, l the slant height,
Lateral or convex surface $=\frac{1}{2}pl$; volume $=\frac{1}{3}a^2h$.

4. In the frustum of a pyramid or cone, if p and q be the perimeters of the two ends; l the slant height; a^2 and b^2 the areas of the two ends; h the perpendicular height,

$$\text{surface} =\frac{l}{2}(p+q);\ \text{volume} =\frac{h}{3}(a^2+ab+b^2).$$

5. In a sphere whose radius is r,
surface $=4\pi r^2$; volume $=\frac{4}{3}\pi r^3$.

6. In the segment of a sphere of which d is the diameter, if h be the height of segment and r the radius of its base;

$$\text{Convex surface} = \pi dh\text{; volume} = \frac{\pi}{6}(3d-2h)h^2\text{, or } \frac{\pi}{6}(3r^2+h^2)h.$$

7. The Imperial gallon $=277{\cdot}274$ cubic inches.

PARALLELOPIPED.

Ex. 8.

1. If the length, breadth, and thickness of a brick be 9, $4\frac{1}{2}$, and 3 inches respectively; find its surface and solidity.

2. How many gallons will a cistern hold, whose length, breadth, and depth are 4 ft. 9 in., 3 ft. 6 in., and 2 ft. 9 in. respectively?

3. A rectangular cistern (open at the top), 8 feet long and 4 feet broad, is made of sheet lead, every square foot of which weighs 16 lbs.; while the whole cistern weighs a ton. How many gallons of water will it hold?

4. A ship's hold is 102 feet long, 40 feet broad, and 5 feet deep; how many bales of goods, each 3 ft. 6 in. long, 2 ft. 3 in. broad, and 2 ft. 6 in. deep, can be stowed into it, leaving a gangway of 4 feet broad?

5. A stone 20 inches long, 15 broad, and 8 deep, weighs 280 lb. How many cubic feet of this kind of stone will freight a vessel of 240 tons burden?

6. A log of timber is 18 feet long, 18 inches broad, and 14 inches thick. If $2\frac{1}{2}$ solid feet be cut off the end of it, what length will be left?

PRISM.

Ex. 9.

1. The length of a triangular prism is 5 feet, and the sides of its base are 6, 8, 10 inches; find the surface and solid content.

2. What is the solid content of the wedge, whose base measures 30 feet by 16 feet, and whose height is 12 feet?

3. Find the content of a prism whose length and perimeter are 15 feet and 35 inches respectively; the base of prism being a regular pentagon.

4. A hexagonal prism is $25\frac{1}{4}$ ft. long, and the central diagonal of its base is $2\frac{1}{4}$ ft.; find the whole surface, and solid content.

PYRAMID.

Ex. 10.

1. Find the whole superficies and the solid content of a triangular pyramid, each side of the base being $5\frac{1}{2}$ feet, and the perpendicular height 30 feet.

Ex. 10.

2. Find the solid content of the frustum of a triangular pyramid, the sides of the base and top being 9, 12, 15 and 6, 8, 10 respectively, and the altitude 20.

3. Find the whole superficies and the solidity of a square pyramid, each side of the base being 12 feet, and the slant height 25 feet.

4. Find the solid content of the frustum of a square pyramid; each side of the greater end being 3 ft. 4 in., and of the less 2 ft. 2 in., the perpendicular height being 10 feet.

5. How many cubic feet of water can be contained in a ditch, of the form of an inverted frustum of a pyramid, if it measure 400 feet by 20 at the top, and 300 feet by 15 at the bottom; the uniform depth being 6 feet?

6. Find the lateral surface and the solid content of a hexagonal pyramid, each side of the base being 2 feet 6 in., and the perpendicular height 10 feet.

7. Find the whole surface and volume of the frustum of an octagonal pyramid, whose perpendicular height is 6 feet, and each side of the two ends 4 feet and 5 feet respectively.

Cylinder.

Ex. 11.

1. What quantity of sheet iron is required to make a funnel 2 feet in diameter, and 40 feet long?

2. What is the solid content of a cylinder, whose diameter is $4\frac{1}{2}$ feet, and height 8 feet?

3. Find the solid content of a cylinder, of which the length is 8 feet 10 inches, and circumference 4 feet 6 inches.

4. The diameter of a well is 3 ft. 9 in., and its depth 45 ft.; what did the excavation cost, at 7*s.* 3*d.* per cubic yard?

Sphere.

Ex. 12.

1. Find the surface and solid content of a sphere, whose diameter is 9 feet.

2. Find the volume of a sphere whose circumference is 45 feet.

3. Find the radius and the circumference of a sphere, whose circumference and solid content have the same numerical value.

4. Suppose the ball on the top of Saint Paul's to be 6 feet in diameter; what would the gilding of it cost at $3\frac{1}{2}$*d.* per square inch?

Ex. 12.

5. If the diameter of the earth be 8000 miles, and geologists knew the interior to the depth of 5 miles below the surface; what fraction of the whole content would be known?

6. Find the convex surface of a slice 2 feet high, cut from a globe of 17 feet radius.

7. Find the surface and solid content of the segment of a sphere, 10 feet in height; the radius of the sphere being 20 feet.

8. The height of a spherical segment is 6 feet, and the circumference of its base 20 feet; find the surface and solid content.

9. If a heavy sphere 4 inches in diameter be placed in a conical glass full of water, whose diameter is 5 and altitude 6 inches; find how much water will run over.

10. What will be the expense of painting the cylindrical pontoon with hemispherical ends, at *6d.* a square yard, the length of the cylindrical part being 19 ft. 4 in., and the common diameter of this cylinder and of the two hemispheres being 2 ft. 8 in.?

11. What will be the weight of a quantity of water equal in bulk to this pontoon, if a cubic foot of water weigh 1000 oz.?

12. If a hemisphere be bisected by a plane parallel to its base, show how to find the height of the segment.

13. Compare the surfaces of the 3 zones of the earth's hemisphere; the torrid zone extending $23\frac{1}{2}°$ from the equator, the frigid zone $23\frac{1}{2}°$ from the pole, and the temperate zone occupying the intermediate space.

14. In a spherical zone, the radii of the two ends are 10 and 6, the altitude 8; find the convex surface and solid content.

15. How much of the earth's surface would a man see, if he were raised to the height of the radius above it? Find also what portion of the volume is contained in the segment seen.

16. To what height must a man be raised above the earth, in order that he may see one-sixth part of its surface? Find also what portion of the volume is contained in the segment.

17. The circumference of the earth being 25000 miles, and the distance between London and York being 200 miles; to what height must a man ascend from one of these places in order that he may see the other?

Cone.

Ex. 13.

1. What quantity of canvas is necessary for a conical tent, whose altitude is 8 feet, and the diameter of the base 13 feet?

Ex. 13.

2. Find the solid content of a cone, the diameter of whose base is $3\frac{1}{2}$ feet, and altitude 6 feet.

3. Find the volume of a cone, when the circumference of the base is 12 feet, and the slant height 15 feet.

4. Find the content of a conic frustum, the circumferences of whose ends are 66 and 56 feet respectively, and the altitude 4 feet.

5. Find the convex surface and the solid content of the frustum of a cone, the perpendicular height of which is 7 feet, and the radii of the two ends 4 feet and 5 feet respectively.

6. If from a right cone whose slant height is 30 feet and circumference of base 10 feet, there be cut off, by a plane parallel to the base, a cone of 6 feet in slant height; what is the convex surface and volume of the frustum?

7. An ale glass in the form of a conic frustum, is $3\frac{5}{8}$ inches in depth, the diameter of the mouth is $2\frac{1}{4}$ in. and that of the bottom 1 in.; find its content, and determine how many of such glasses an imperial gallon would be equivalent to.

8. A cask, in the form of 2 conic frustums joined at the bases, has the diameter at the head 20 inches and at the bung 25 in.; also the length is 3 feet 4 inches; find the weight of water required to fill it, supposing that a cubic foot of water weighs 1000 oz.

9. A right cone and a hemisphere lie on opposite sides of a common base of 2 feet diameter, and the cone is right-angled at the vertex. If a cylinder circumscribe them in this position, how much additional space is thereby enclosed?

Artificers' Work.

The standard thickness of brick-work is 3 half-bricks; any other thickness must be reduced to this standard by multiplying the superficial content of the wall by the number of half-bricks, and dividing the product by 3.

The standard rod of brick-work contains $272\frac{1}{4}$ square feet.

Ex. 14.

1. How many standard rods of brick-work are there in a wall 60 feet long, 12 feet high, and 3 bricks thick?

2. A triangular gable 18 feet high, of 1 brick thick, is raised on an end wall 20 feet long and 30 feet high, of 2 bricks thick; what is the cost of the whole at £4 a standard rod?

3. What will it cost to build a wall 12 feet high, and 2 bricks

Ex. 14.

thick, round a garden of rectangular form, which contains 3 roods, and of a length equal to twice its breadth, at £5 a standard rod?

4. What is the expense of tiling a house at 25*s.* per square of 100 feet, the length of the house being 50 feet, and the breadth 30 feet; the girt over being $\frac{3}{2}$ of the breadth of the house, and the eaves which project 1 foot on each side being reckoned into the work?

5. What does the wainscoting of a room cost, at £3 15*s.* a square of 100 feet, if the length, breadth and height be respectively 21, 15 and 10 feet; the door, which measures 6 ft. by 4 ft., and two window shutters, each 5 ft. by 4 ft., being reckoned work and half work?

6. Find the thickness of lead, in the pipe of $1\frac{1}{4}$ in. bore, which weighs 14 lb. per yard in length.

N.B. A cubic foot of lead is supposed, in Exs. 6, 7, to weigh 11325 *oz.*

7. What is the expense of a leaden pipe of 2 inches bore, half an inch thick, and 4 yards long, at $2\frac{1}{2}$*d.* a pound?

MEASUREMENT OF SHOT, SHELLS AND POWDER*.

In the following questions it is assumed—

1. That an iron ball of 4 inches diameter weighs 9 lb.
2. That a leaden ball of 1 inch diameter weighs $\frac{3}{14}$ lb.
3. That 30 cubic inches of gunpowder weigh 1 lb.

Ex. 15.

1. What is the weight of an iron ball of 1 foot diameter?

2. What is the diameter of an iron ball which weighs 200 lb.?

3. What is the weight of an iron shell, the external and internal diameters of which are 9 inches and 6 inches respectively?

4. If the outer diameter of a shell that weighs 75 lb. be 9 inches; what is the inner diameter?

5. What is the weight of a leaden ball of 3 inches diameter?

6. What is the diameter of a leaden ball which weighs 4 lb.?

7. Show that a 21 lb. iron ball is of the same size as a 32 lb. leaden ball.

8. What weight of powder will fill a box, whose three dimensions are 2 feet, $1\frac{1}{2}$ feet, and 1 foot?

9. How many inches are there in each side of a cubical box that holds 100 lb. of powder?

* For Examples on the Piling of Balls and Shells, see pp. 56 & 57.

Ex. 15.

10. How much powder will fill a shell, whose internal diameter is 9 inches?

11. What is the internal diameter of a shell, that holds 10 lb. of powder?

12. What weight of powder will fill a cylinder whose height is 3 feet, and diameter of base 10 inches?

13. What length of a gun of 6 inches bore will be filled with 10 lb. of powder?

14. What is the radius of the base of a cylinder 4 feet high, that holds 100 lb. of powder?

15. What weight of powder will fill a cone, whose altitude is 7 feet, and diameter of base 2 feet?

16. What is the altitude of a cone, the radius of whose base is 9 inches, that holds 50 lb. of powder?

17. What is the radius of the base of a cone, whose altitude is 8 feet, and which holds 450 lb. of powder?

18. What is the calibre of a gun, that carries a leaden ball of 4 ounces weight, allowing $\frac{1}{49}$th of the ball's diameter for windage?

PLANE TRIGONOMETRY.

TRIGONOMETRICAL FORMULÆ.

Ex. 1.

1. Express the angles $15° 4' 17''$, $113° 0' 35''$, and $78° 50' 41''\cdot 3$; according to the French or centesimal division of the circle.

2. Find the complement of the angles—
$34° 15' 7''$; $159° 18' 0''\cdot 7$; $225° 31' 28''$.

3. Find the supplement of the angles—
$54° 10' 56''$; $144° 37' 41''$; $314° 10' 17''$.

4. Find the circular measure of the angles—
$57°$; $57° 30'$; $18°$; $54°$; $22° 30'$.

5. Find the angles whose circular measures are—
1; $\frac{1}{2}$; $\frac{3}{4}$; $\frac{1}{7}$; $\frac{2}{7}$; $\frac{5}{6}$.

6. If $\cos A = \cdot 6$; find $\sin A$, $\cot A$, and chd$A$.

7. If $\sin A = \frac{1}{2}$; find $\sin 3A$, $\cos 4A$, and $\tan 2A$.

8. If $\cot A = \frac{4}{3}$; find cosec $2A$, and versin A.

9. If $\tan \frac{A}{2} = 2 - \sqrt{3}$; find $\sin A$.

Ex. 2. Prove the formulæ—

1. $\cot A + \tan A = 2 \operatorname{cosec} 2A$.

2. $\operatorname{cosec} 2A + \cot 2A = \cot A$.

3. $\sec A = 1 + \tan A \tan \frac{A}{2}$.

4. $\operatorname{cosec} 2A = \frac{1 + \cot^2 A}{2 \cot A}$.

5. $\operatorname{versin} A = \tan \frac{A}{2} \sin A$.

6. $\cos A = \cos^4 \frac{A}{2} - \sin^4 \frac{A}{2}$.

7. $2 \operatorname{cosec} 2A = \sec A \operatorname{cosec} A$.

8. $\cot^2 A \cos^2 A = \cot^2 A - \cos^2 A$.

9. $\tan A + \sec A = \tan \left(45° + \frac{A}{2}\right)$.

10. $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$.

11. $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

12. $\dfrac{\cos^2 A - \sin^2 B}{\sin^2 A \sin^2 B} = \cot^2 A \cot^2 B - 1$.

Ex. 2. Prove the formulæ—

13. $\tan 2A - \tan A = \dfrac{2 \sin A}{\cos A + \cos 3A}$.

14. $\tan^2 2A - \tan^2 A = \dfrac{\sin 3A \sin A}{\cos^2 2A \cos^2 A}$.

15. $\dfrac{\sin 2A}{1 + \cos 2A} \times \dfrac{\cos A}{1 + \cos A} = \tan \dfrac{A}{2}$.

16. $\dfrac{2 \sin A + \sin 2A}{2 \sin A - \sin 2A} = \cot^2 \dfrac{A}{2}$.

17. $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$.

18. $\sin 5A \sin A = \sin^2 3A - \sin^2 2A$.

19. $\sin 7A \sin 3A = \sin^2 5A - \sin^2 2A$.

20. $\sin 3A \sin^3 A + \cos 3A \cos^3 A = \cos^3 2A$.

21. $\tan 3A \tan A = \dfrac{\cos 2A - \cos 4A}{\cos 2A + \cos 4A}$.

22. $\dfrac{1 - 2 \sin^2 A}{1 + \sin 2A} = \dfrac{1}{\sec 2A + \tan 2A}$.

23. $\cos A(1 - \tan 2A \tan A) = \cos 3A(1 + \tan 2A \tan A)$.

24. $\sin A(\tan A + 2 \cot 2A) = \cos^2 A\left(1 + \tan A \tan \dfrac{A}{2}\right)$.

25. $\tan^4 A = \dfrac{\sin^2 2A - 4 \sin^2 A}{\sin^2 2A + 4 \sin^2 A - 4} = \dfrac{\cos^2 2A - 4 \cos^2 A + 3}{\cos^2 2A + 4 \cos^2 A - 1}$.

26. $2 + 4 \cot^2 2A = \tan^2 A + \cot^2 A$.

27. $\left(1 + \tan \dfrac{A}{2} + \sec \dfrac{A}{2}\right)\left(1 + \tan \dfrac{A}{2} - \sec \dfrac{A}{2}\right) = \sin A \sec^2 \dfrac{A}{2}$.

28. $\left(\cot \dfrac{A}{2} - \tan \dfrac{A}{2}\right)^2 (1 - 2 \tan A \cot 2A) = 4$.

29. $\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B$.

30. $\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$.

31. $2(\sin^2 A \sin^2 B + \cos^2 A \cos^2 B) = 1 + \cos 2A \cos 2B$.

32. $\tan (A + B) = \dfrac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B}$.

33. $\cos A + \cos (A + 2B) = 2 \cos (A + B) \cos B$.

34. $\sin (A - B) \sin C - \sin (A - C) \sin B + \sin (B - C) \sin A = 0$.

Ex. 2. Prove the formulæ—

35. $\sin(A+B)\cos B - \sin(A+C)\cos C = \sin(B-C)\cos(A+B+C)$.

36. $\sin(A+B-2C)\cos B - \sin(A+C-2B)\cos C$
$= \sin(B-C)\{\cos(B+C-A) + \cos(A+C-B) + \cos(A+B-C)\}$.

37. $\sin(A+B)\sin(B+C+D)$
$= \sin A \sin(C+D) + \sin B \sin(A+B+C+D)$.

38. $4\sin(A-B)\sin(mA-B)\cos(m-1)A$
$= 1 + \cos 2(m-1)A - \cos 2(A-B) - \cos 2(mA-B)$.

39. $4\cos mA \cos nA \cos rA = \cos(m+n+r)A + \cos(m+n-r)A$
$+ \cos(m-n+r)A + \cos(m-n-r)A$.

40. $\sin A + \sin B + \sin C = 4\sin\frac{A+B}{2}\sin\frac{A+C}{2}\sin\frac{B+C}{2} + \sin(A+B+C)$.

41. $\cos A + \cos B + \cos C = 4\cos\frac{A+B}{2}\cos\frac{A+C}{2}\cos\frac{B+C}{2} - \cos(A+B+C)$.

42. $\tan A + \tan B + \tan C = \tan A \tan B \tan C + \frac{\sin(A+B+C)}{\cos A \cos B \cos C}$.

43. $\cot A + \cot B + \cot C = \cot A \cot B \cot C - \frac{\cos(A+B+C)}{\sin A \sin B \sin C}$.

44. $4\sin\frac{1}{2}(A+B+C)\sin\frac{1}{2}(B+C-A)\sin\frac{1}{2}(A+C-B)\sin\frac{1}{2}(A+B-C)$
$= 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2\cos A \cos B \cos C$.

45. $4\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)\cos\frac{1}{2}(A+C-B)\cos\frac{1}{2}(A+B-C)$
$= -1 + \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C$.

46. $\operatorname{vers}(A-B)\operatorname{vers}\{180^\circ - (A+B)\} = (\sin A - \sin B)^2$.

47. $\operatorname{chd}\frac{3A}{2}\operatorname{chd}\frac{A}{2} = \operatorname{chd}^2 A - \operatorname{chd}^2\frac{A}{2}$.

48. $\frac{\sin A}{\sin B \cos B} - \frac{\cos(A-B)}{\cos B} - \frac{\sin A}{\sin B} + 1 = \frac{2\left(\sin B \sin^2\frac{A}{2} - \sin A \sin^2\frac{B}{2}\right)}{\sin B}$.

Ex. 3. Find the values of—

1.	$\sin 7^\circ 30'$;	$\sin 9^\circ$;	$\sin 22^\circ 30'$;	$\sin 747^\circ$.
2.	$\cos 12^\circ$;	$\cos 11^\circ 15'$;	$\cos 33^\circ 45'$;	$\cos 549^\circ$.
3.	$\tan 9^\circ$;	$\tan 22^\circ 30'$;	$\tan 37^\circ 30'$;	$\tan 165^\circ$.
4.	$\cot 18^\circ$;	$\cot 75^\circ$;	$\cot 225^\circ$;	$\cot 330^\circ$.
5.	$\sec 22^\circ 30'$;	$\sec 54^\circ$;	$\sec 225^\circ$;	$\sec 195^\circ$.
6.	$\operatorname{cosec} 60^\circ$;	$\operatorname{cosec} 72^\circ$;	$\operatorname{cosec} 216^\circ$;	$\operatorname{cosec} 387^\circ$.

Ex. 3. Find the values of—

7. vers $15°$; vers $67° 30'$; vers $240°$; vers $342°$.
8. chd $36°$; chd $45°$; chd $240°$; chd $288°$.

Ex. 4. Prove the formulæ—

1. $\sin 3A = 4 \sin A \sin (60° + A) \sin (60° - A)$.
2. $2 \sec 2A = \sec (45° + A) \sec (45° - A)$.
3. $\tan\left(45° + \frac{A}{2}\right) + \cot\left(45° + \frac{A}{2}\right) = 2 \sec A$.
4. $\sin 2A = \frac{1 - \cot^2 (45° + A)}{1 + \cot^2 (45° + A)}$.
5. $\frac{\sin 60° - \sin 30°}{\sin 60° + \sin 30°} = \frac{\tan 60° - \tan 45°}{\tan 60° + \tan 45°}$.
6. $\text{chd}^2 (90° - A) = 2 - 2 \sin A$.
7. $\text{chd } 108° = \text{chd } 36° + \text{chd } 60°$.

Ex. 5. If $A + B + C = 90°$; prove that—

1. $\tan A \tan B + \tan A \tan C + \tan B \tan C = 1$.
2. $\cot A + \cot B + \cot C = \cot A \cot B \cot C$.
3. $\tan A + \tan B + \tan C = \tan A \tan B \tan C + \sec A \sec B \sec C$.
4. $\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C$.
5. $\frac{\cos A + \sin C - \sin B}{\cos B + \sin C - \sin A} = \frac{1 + \tan \frac{1}{2}A}{1 + \tan \frac{1}{2}B}$.

Ex. 6. If $A + B + C = 180°$; prove that—

1. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
2. $\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C + 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = 1$.
3. $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$.
4. $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.
5. $\cos A + \cos B + \cos C = 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C + 1$.
6. $1 + \cos A \cos B \cos C = \cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin A \sin B$.
7. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
8. $\cot A \cot B + \cot A \cot C + \cot B \cot C = 1$.
9. $\cot A + \cot B + \cot C = \cot A \cot B \cot C + \text{cosec} A \,\text{cosec} B \,\text{cosec} C$.
10. If $\sin^3 \theta = \sin (A - \theta) \sin (B - \theta) \sin (C - \theta)$; show that
$\cot \theta = \cot A + \cot B + \cot C$; $\text{cosec}^2 \theta = \text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C$.

Ex. 7.

1. If in any triangle ABC, $\sin A$, $\sin B$, $\sin C$ be in Ar. Prog. prove that $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are also in Ar. Prog.
2. If $\theta-\alpha$, θ, $\theta+\alpha$ be three angles whose cosines are in Harmonical progression; show that $\cos\theta=\sqrt{2}\cos\frac{\alpha}{2}$.
3. If $\cos(A-C)\cos B=\cos(A-B+C)$; prove that $\tan A$, $\tan B$, $\tan C$ are in Harmonical progression.

Prove that—

4. If $1=\left(\frac{\sin A}{\sin B}\right)^2+(\cos A\cos C)^2$; then $\sin C=\frac{\tan A}{\tan B}$.
5. If $\tan A=\frac{b\sin B}{a+b\cos B}$; then $\tan(B-A)=\frac{a\sin B}{b+a\cos B}$.
6. If $\sin B=m\sin(2A+B)$; then $\tan(A+B)=\frac{1+m}{1-m}\tan A$.
7. If $\cos A=\cos B\cos C$; $\tan\frac{1}{2}(A+B)\tan\frac{1}{2}(A-B)=\tan^2\frac{1}{2}C$.

Ex. 8. Determine A in the following equations—

1. $\sin A=\sin 2A$.
2. $2\sin A=\tan A$.
3. $\tan 2A=3\tan A$.
4. $\tan A=\operatorname{cosec} 2A$.
5. $2\sin^2 3A+\sin^2 6A=2$.
6. $\sin^2 2A-\sin^2 A=\frac{1}{4}$.
7. $\tan A+3\cot A=4$.
8. $\tan^2 A+4\sin^2 A=6$.
9. $\tan A+\cot A=4$.
10. $2\cos 2A=2\sin A+1$.
11. $\cos 3A+\cos 2A+\cos A=0$.
12. $4\sin A\sin 3A=1$.
13. $\tan A+2\cot 2A=\sin A\left(1+\tan A\tan\frac{A}{2}\right)$.

Ex. 9. Determine x in the following equations—

1. $\sin(x+\alpha)=\cos(x-\alpha)$.
2. $\sqrt{2}(\cos 3x+\sin 3x)=1$.
3. $\sin 7x-\sin x=\sin 3x$.
4. $\sin(x+\alpha)+\cos(x+\alpha)=\sin(x-\alpha)+\cos(x-\alpha)$.
5. $\tan(x+\alpha)\tan(x-\alpha)=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha}$.
6. $\tan\alpha\tan x=\tan^2(\alpha+x)-\tan^2(\alpha-x)$.
7. $\{(1+\sin x)^{\frac{1}{2}}-1\}\{(1-\sin x)^{\frac{1}{2}}+1\}=\tan\frac{1}{2}\alpha\sin x$.

Ex. 9. Determine x in the following equations—

8. $\sin x \sin(2\alpha+x)+n\cos^2\alpha=0$.

9. $\tan^3 x=\tan(x-\alpha)$.

10. $\sin\alpha+\sin(x-\alpha)+\sin(2x+\alpha)=\sin(x+\alpha)+\sin(2x-\alpha)$.

Ex. 10. Prove that—

1. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}=90°$.

2. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=45°$.

3. $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=45°$.

4. $\cot^{-1}\frac{3}{4}+\cot^{-1}\frac{1}{7}=135°$.

5. $\sin^{-1}\dfrac{1}{\sqrt{5}}+\cot^{-1}3=45°$.

6. $\cos^{-1}\left(\dfrac{2}{3}\right)^{\frac{1}{2}}-\cos^{-1}\left(\dfrac{1}{\sqrt{2}}+\dfrac{\sqrt{3}}{6}\right)=30°$.

7. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=45°$.

8. $\tan^{-1}\dfrac{t_1-t_2}{1+t_1t_2}+\tan^{-1}\dfrac{t_2-t_3}{1+t_2t_3}+\&c.+\tan^{-1}\dfrac{t_{n-1}-t_n}{1+t_{n-1}t_n}=\tan^{-1}t_1-\tan^{-1}t_n$

9. $\sin^{-1}\left(\dfrac{x}{a+x}\right)^{\frac{1}{2}}=\tan^{-1}\left(\dfrac{x}{a}\right)^{\frac{1}{2}}$.

10. $\tan^{-1}\left(\dfrac{x\cos\phi}{1-x\sin\phi}\right)-\tan^{-1}\left(\dfrac{x-\sin\phi}{\cos\phi}\right)=\phi$.

Ex. 11.

1. If $\theta=\tan^{-1}\dfrac{1}{\sqrt{3}}$, $\phi=\tan^{-1}\dfrac{1}{\sqrt{15}}$; $\sin(\phi+\theta)=\sin 60°\cos 36°$.

2. If $\cot^{-1}(x-1)-\cot^{-1}(x+1)=\dfrac{\pi}{12}$; find x.

3. If $\sec^{-1}a+\sec^{-1}\dfrac{x}{a}=\sec^{-1}b+\sec^{-1}\dfrac{x}{b}$; find x.

4. If $\text{vers}^{-1}(1+x)-\text{vers}^{-1}(1-x)=\tan^{-1}2(1-x^2)^{\frac{1}{2}}$; find x.

5. If $\cos v=\dfrac{\cos u-e}{1-e\cos u}$; find $\tan\frac{1}{2}v$ in terms of $\tan\frac{1}{2}u$.

6. If $\tan\theta=\dfrac{b}{a}$; find the value of $a\cos 2\theta+b\sin 2\theta$.

7. If $\tan\theta=\left(\dfrac{b}{a}\right)^{\frac{1}{3}}$; find the value of $\dfrac{a}{\cos\theta}+\dfrac{b}{\sin\theta}$.

8. If $\sin\theta=\sin\alpha\sin(\phi+\theta)$; find $\tan\theta$ in terms of α and ϕ.

9. If $\tan\theta=\cos\alpha\tan\phi$; find $\tan(\phi-\theta)$ in terms of α and ϕ.

Ex. 11.

10. If $\tan\theta=\tan^3(\frac{1}{2}\phi)$, $\cos^2\phi=\frac{1}{3}(m^2-1)$; find m in terms of θ.
11. If $\tan\frac{1}{2}\alpha=\tan^3(\frac{1}{2}\beta)$, $\tan\beta=2\tan\phi$; prove that $2\phi=\alpha+\beta$.
12. If $\tan(45^\circ-\frac{1}{2}\theta)=\tan^3\phi$; prove that
$$(\tan\theta+\sec\theta)^{\frac{1}{3}}+(\tan\theta-\sec\theta)^{\frac{1}{3}}=2\cot 2\phi.$$
13. If $\cos\theta=\cos^2\alpha-\sin^2\alpha(1-c^2\sin^2\theta)^{\frac{1}{2}}$; prove that
$$\tan\frac{\theta}{2}=\tan\alpha(1-c^2\sin^2\alpha)^{\frac{1}{2}}.$$

Ex. 12.

1. Eliminate θ from the equations,
$$m=\operatorname{cosec}\theta-\sin\theta,\quad n=\sec\theta-\cos\theta.$$
2. Eliminate θ from the equations,
$$(a+b)\tan(\theta-\phi)=(a-b)\tan(\theta+\phi),\quad a\cos 2\phi+b\cos 2\theta=c.$$
3. Eliminate θ and ϕ from the equations,
$$\cos^2\theta=\frac{\cos\alpha}{\cos\beta},\quad \cos^2\phi=\frac{\cos\gamma}{\cos\beta},\quad \frac{\tan\theta}{\tan\phi}=\frac{\tan\alpha}{\tan\gamma}.$$
4. Eliminate θ and ϕ from the equations,
$$a\sin^2\theta+b\cos^2\theta=m,\quad b\sin^2\phi+a\cos^2\phi=n,\quad a\tan\theta=b\tan\phi.$$
5. Eliminate x, y, z from the equations,
$$b^2(x^2-y^2)\cos\theta=a^2z^2\cos\phi;\quad \frac{\sin(\theta+\phi)}{x}=\frac{\sin(\theta-\phi)}{y}=\frac{\sin 2\theta}{z}.$$
6. Eliminate a and b from the equations,
$$a^2+b^2=\frac{V^2}{\mu}+R^2,\quad ab=\frac{VR}{\mu^{\frac{1}{2}}}\sin\alpha,\quad \left(\frac{R\cos\omega}{a}\right)^2+\left(\frac{R\sin\omega}{b}\right)^2=1.$$

PROPERTIES OF PLANE FIGURES.

Ex. 13. In any right-angled triangle ABC, C being the right angle, and a, b, c the sides opposite to the angles A, B, C respectively; prove that—

1. $\cos(A-B)=\dfrac{2ab}{c^2}$.
2. $\cos(2A-B)=\dfrac{a}{c^3}(3c^2-4a^2)$.
3. $\cos 2A=\dfrac{b^2-a^2}{b^2+a^2}$.
4. $\cos(45^\circ\pm A)=\dfrac{1}{\sqrt{2}}\left(\dfrac{b\mp a}{c}\right)$.
5. $\tan 2A=\dfrac{2ab}{b^2-a^2}$.
6. $\tan\frac{1}{2}A=\left(\dfrac{c-b}{c+b}\right)^{\frac{1}{2}}$.
7. Area $=\dfrac{(a+b+c)(a+b-c)}{4}$; $=\dfrac{c^2}{4}\sin 2A$; $=\dfrac{b^2}{2}\tan A$.
8. $R+r=\frac{1}{2}(a+b)$; R, r being the radii of circles, one described about, the other inscribed in, the right-angled triangle.

Ex. 14. In any triangle ABC, having a, b, c sides opposite to the angles A, B, C respectively, prove that—

1. $a = b\cos C + c\cos B.$
2. $\tan B = \dfrac{b\sin C}{a - b\cos C}.$
3. $\dfrac{\tan B}{\tan C} = \dfrac{a^2+b^2-c^2}{a^2+c^2-b^2}.$
4. $\dfrac{\text{vers}\, A}{\text{vers}\, B} = \dfrac{a(a+c-b)}{b(b+c-a)}.$
5. $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{a^2-b^2}{c^2}.$
6. $\cos A + \cos B = \dfrac{a+b}{c}\cdot(2\sin^2\tfrac{1}{2}C).$
7. $\sin\tfrac{1}{2}(A-B) = \dfrac{a-b}{c}\cdot\cos\tfrac{1}{2}C.$
8. $\cos\tfrac{1}{2}(A-B) = \dfrac{a+b}{c}\cdot\sin\tfrac{1}{2}C.$
9. $\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B = \dfrac{a+b-c}{a+b+c}.$
10. $\dfrac{\tan\frac{1}{2}A + \tan\frac{1}{2}B}{\tan\frac{1}{2}A - \tan\frac{1}{2}B} = \dfrac{c}{a-b}.$
11. $\dfrac{\sin^2\frac{1}{2}A}{\sin^2\frac{1}{2}C} = \dfrac{a(a+b-c)}{c(b+c-a)}.$
12. $\dfrac{\cot\frac{1}{2}B + \cot\frac{1}{2}C}{\cot\frac{1}{2}A} = \dfrac{2a}{b+c-a}.$
13. $\tfrac{1}{2}(a^2+b^2+c^2) = ab\cos C + ac\cos B + bc\cos A.$
14. $\text{Area} = \tfrac{1}{2}ab\sin C; \ = \tfrac{1}{2}c^2\dfrac{\sin A\sin B}{\sin C}; \ = \tfrac{1}{2}(a^2-b^2)\dfrac{\sin A\sin B}{\sin(A-B)}.$
15. $\text{Area} = \dfrac{a^2+b^2-c^2}{4\tan\frac{1}{2}(A+B-C)}; \ = \dfrac{2abc}{a+b+c}\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$
16. $\text{Area} = \tfrac{1}{4}(a+b+c)^2.\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C.$

If R, r be the radii of circles, respectively described about, and inscribed in, any triangle; prove that—

17. $\dfrac{1}{R} = \dfrac{8\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{a+b+c}; \ = 2\left(\dfrac{\sin A\sin B\sin C}{abc}\right)^{\frac{1}{2}}.$
18. $r = \tfrac{1}{2}(a+b+c)\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C; \quad 2Rr = \dfrac{abc}{a+b+c}.$
19. $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C.$
20. If α, β, γ be the angles, which the sides of a triangle subtend at the centre of the inscribed circle; show that
$$4\sin\alpha\sin\beta\sin\gamma = \sin A + \sin B + \sin C.$$

Ex. 15.

1. If the sides a, b of a triangle include an angle of 120°, show that $c^2 = a^2 + ab + b^2$.
2. If $2\cos B = \dfrac{\sin A}{\sin C}$, the triangle is isosceles.
3. If $\dfrac{\tan A}{\tan B} = \dfrac{\sin^2 A}{\sin^2 B}$, the triangle is isosceles or right-angled.

Ex. 15.

4. The vertical angle of an isosceles triangle is 90°; $2p$ is the perimeter of the triangle and r the radius of the inscribed circle; show that $p : r = 2^{\frac{1}{2}} + 1 : 2^{\frac{1}{2}} - 1$.

5. If a line CE bisect the angle C of any triangle and meet the base in E; $\tan \text{AEC} = \dfrac{a+b}{a-b} \tan \frac{1}{2}\text{C}$; $\text{CE} = \dfrac{2ab}{a+b} \cos \frac{1}{2}\text{C}$.

6. Prove that the distance of the centre of the circle inscribed in any triangle ABC from A is equal to $\dfrac{2bc}{a+b+c} \cos \frac{1}{2}\text{A}$.

7. Having given the perimeter $2p$, and the three angles of a triangle; find the sides a, b, c.

8. If, in a triangle, the angles are such, that $\text{A} : \text{B} : \text{C} = 2 : 3 : 4$; then will $\cos \frac{1}{2}\text{A} = \dfrac{a+c}{2b}$.

9. If, in a triangle ABC, the angles are such, that $\text{A} = 2\text{B} = 2^2\text{C}$; show that the side, $a = 2(a+b+c) \sin 12^\circ\frac{6}{7}$.

10. If, for any triangle, $x + \dfrac{1}{x} = 2 \cos \text{A}$, and $y + \dfrac{1}{y} = 2 \cos \text{B}$; show that $bx + \dfrac{a}{y} = c$.

11. In any triangle, the length of a perpendicular from A on the opposite side $= \dfrac{b^2 \sin \text{C} + c^2 \sin \text{B}}{b+c}$.

12. In any triangle the distance of a perpendicular drawn from C on AB, from the middle point of $\text{AB} = \dfrac{c}{2} \cdot \dfrac{\tan \text{A} - \tan \text{B}}{\tan \text{A} + \tan \text{B}}$.

13. If, in a triangle ABC, $b - a = nc$; show that
$$\cos\left(\text{A} + \frac{\text{C}}{2}\right) = n \cos \frac{\text{C}}{2}, \text{ and } \cot \frac{\text{B}-\text{A}}{2} = \frac{1 + n \cos \text{B}}{n \sin \text{B}}.$$

14. Given the three straight lines drawn from any point to the three angular points of an equilateral triangle; find one of its sides.

15. If through a point O within a triangle, three straight lines be drawn from the angles A, B, C, meeting the opposite sides in D, E, F respectively; then $\dfrac{\text{OD}}{\text{AD}} + \dfrac{\text{OE}}{\text{BE}} + \dfrac{\text{OF}}{\text{CF}} = 1$.

16. If a straight line intersect the two sides AC, BC of a plane triangle in the points b, a, and the base AB produced in c; then $\text{A}b \,.\, \text{B}c \,.\, \text{C}a = \text{A}c \,.\, \text{B}a \,.\, \text{C}b$.

Ex. 15.

17. The sides of a triangle are 3, 5, 6; find the radii of the inscribed and circumscribed circles.

18. Two sides of a triangle are 3 and 12, and the contained angle is 30°; find the hypothenuse of an equivalent right-angled isosceles triangle.

19. In a triangle, having given B, a, and its area, construct the triangle.

20. Having given A, B, C the angles of a triangle, and R the radius of the circumscribing circle; find a, b, c.

21. If in a right-angled triangle, a perpendicular be drawn from the right angle to the hypothenuse; the areas of the two circles inscribed in the triangles on each side of this perpendicular are proportional to the corresponding segments of the hypothenuse.

22. If r be the radius of the circle inscribed in a triangle whose sides are a, b, c; and h, k, l be the distances of its centre from the angles of the triangle; show that $\frac{hkl}{r}=\frac{2abc}{a+b+c}$.

23. If r_1, r_2, r_3 be the radii of circles which touch respectively a side of a triangle and the other two sides produced, prove that

$$r_1r_2r_3=abc\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

24. If r be the radius of the circle inscribed in a triangle, and r_1, r_2, r_3 the radii of the circles inscribed between this circle and the sides containing the angles A, B, C respectively; prove that

$$(r_1r_2)^{\frac{1}{2}}+(r_1r_3)^{\frac{1}{2}}+(r_2r_3)^{\frac{1}{2}}=r.$$

25. If r_0 denote the radius of the circle inscribed in any triangle; r_1, r_2, r_3 the radii of circles which touch each side respectively, and the other two produced; show that

$$\frac{1}{r_0}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3};\ \text{area of triangle}=(r_0r_1r_2r_3)^{\frac{1}{2}}.$$

26. The area of any triangle is to the area of the triangle whose sides are respectively equal to the lines joining its angular points with the middle points of the opposite sides as 4 : 3.

27. The area of any triangle is to the area of the triangle formed by joining the points where the lines bisecting the angles meet the opposite sides as $(a+b)(a+c)(b+c)$: $2abc$.

28. The sides of a triangle are in Arithmetical progression, and its area is to the area of an equilateral triangle of the same perimeter as 3 : 5; find the ratio of the sides, and the value of the greatest angle.

29. Find the ratio between (1) the sides, (2) the areas, of an equilateral triangle and a square inscribed in the same circle.

Ex. 15.

30. Compare the areas (1) of regular pentagons, (2) of regular octagons described within and about a circle.

31. The square of a side of the regular pentagon inscribed in a circle is equal to the square of a side of the inscribed hexagon, together with the square of a side of the inscribed decagon.

32. If, in a regular polygon of n sides, each side is $2a$; the sum of the radii of the circumscribed and inscribed circles is $a \cot \frac{90^\circ}{n}$.

33. The area of a regular polygon inscribed in a circle is a mean proportional between the areas of an inscribed and a circumscribed regular polygon of half the number of sides.

34. A, B, C, are 3 regular octagons; a side of A is equal to the diameter of the circle inscribed in B, or described about C; find the sum of the three areas, that of A being a^2.

35. The distance between the centres of two wheels is a, and the sum of their radii is c; find the length of a string which crosses between them and just wraps round them.

36. If two circles, whose radii are a, b, touch one another externally, and if θ be the angle contained by the two common tangents to these circles; show that $\sin\theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}$.

TRIGONOMETRICAL TABLES.

Ex. 16. Prove that—

1. $\sin 123^\circ\, 14'\, 20'' = \cdot 836393$; $\cos 41^\circ\, 13'\, 26'' = \cdot 752140$.
2. $\tan 28^\circ\, 13'\, 47'' = \cdot 536862$; $\cot 15^\circ\, 16'\, 45'' = 3\cdot 660643$.
3. $\sec 80^\circ\, 59'\, 30'' = 6\cdot 386593$; $\operatorname{cosec} 100^\circ\, 10'\, 10'' = 1\cdot 015979$.

Determine A in the equations—

4. $\sin A = \cdot 346105$; $\cos A = \cdot 938629$.
5. $\sec A = 2\cdot 005263$; $\operatorname{cosec} A = 4\cdot 257000$.

Prove that—

6. $\log\sin 15^\circ 16' 17'' = 9\cdot 420601$; $\log\cos 32^\circ\, 14'\, 55'' = 9\cdot 927237$.
7. $\log\tan 23^\circ 24' 25'' = 9\cdot 636370$; $\log\cot 53^\circ\, 14'\, 15'' = 9\cdot 873364$.
8. $\log\sec 75^\circ 13' 40'' = 10\cdot 593499$; $\log\operatorname{cosec} 130^\circ\, 29'\, 30'' = 10\cdot 118900$.

Ex. 16. Determine A in the equations—

9. $\log \sin A=9{\cdot}246179$; $\log \cos A=9{\cdot}123456$.
10. $\log \tan A=11{\cdot}012440$; $\log \cot A=9{\cdot}876543$.
11. $\log \sec A=10{\cdot}124625$; $\log \operatorname{cosec} A=11{\cdot}024981$.

SOLUTION OF PLANE TRIANGLES.

Ex. 17. Solve the triangle ABC, when the parts given are—

1. $C=90^\circ$. $A=18^\circ\,14'$. $AB=432$.
2. $C=90^\circ$. $A=42^\circ\,35'$. $BC=296{\cdot}35$.
3. $C=90^\circ$. $B=25^\circ\,19'\,13''$. $AB=1234$.
4. $C=90^\circ$. $B=56^\circ\,53'\,35''$. $AC=184$.
5. $C=90^\circ$. $AB=957{\cdot}34$. $AC=24$.
6. $C=90^\circ$. $AB=3562$. $BC=357$.
7. $C=90^\circ$. $AC=423$. $BC=584$.
8. $A=31^\circ\,13'$. $B=48^\circ\,24'\,15''$. $BC=926{\cdot}7$.
9. $B=39^\circ\,15'$. $C=13^\circ$. $AC=326$.
10. $AB=527$. $AC=635{\cdot}91$. $B=88^\circ\,14'$.
11. $AC=333$. $BC=325$. $A=52^\circ\,19'$.
12. $AB=1249{\cdot}6$. $BC=397{\cdot}3$. $A=8^\circ\,19'\,35''$.
13. $C=124^\circ\,34'$. $AC=126$. $AB=264$.
14. $A=53^\circ\,14'$. $AB=521{\cdot}05$. $BC=417{\cdot}4026$.
15. $AB=426$. $AC=354$. $A=49^\circ\,16'$.
16. $AB=53{\cdot}94$. $BC=156{\cdot}5$. $B=15^\circ\,13'\,14''$.
17. $AB=1263$. $AC=1359$. $BC=1468$.
18. $AB=12{\cdot}6356$. $AC=13{\cdot}4837$. $BC=11{\cdot}2983$.
19. Find the angles of a triangle ABC when $4AB=7AC$ and $A=9^\circ\,15'\,35''$.

HEIGHTS AND DISTANCES.

Ex. 18.

1. At 120 feet distance from the foot of a steeple, the angle of elevation of the top was found to be $60^\circ\,30'$. Required the height.

2. From the top of a rock 326 feet above the sea, the angle of depression of a ship's bottom was found to be 24°. Required the distance of the ship.

3. A wall is surrounded by a ditch; from the edge of this ditch the angle of elevation of a point on the top of the wall is found to be 35°; and at a distance of 100 yards from the ditch the angle of elevation of the same point is found to be 15°. Find the height of the wall; the breadth of the ditch; and the length of the ladder that would just reach from the edge of the ditch to the top of the wall.

4. From the top of a hill I observed two milestones in a straight line before me; and found their angles of depression to be 5° and 15°, what is the height of the hill?

5. Two observers, on the same side of a balloon, in the same vertical plane with it, and a mile apart, find its angles of elevation to be 15° and 65° 30′. Find the height of the balloon.

6. A ladder 38 feet long, just reaches to a window 29 feet 6 inches high on one side of a street; and, on turning the ladder over without moving its foot, it reaches a window 28 feet high on the other side. Find the breadth of the street.

7. The top of a maypole being broken off, struck the ground at a distance of $13\frac{3}{8}$ feet from the bottom of the pole; and the broken piece was found to measure $29\frac{4}{5}$ feet. Find the original height of the pole.

8. The aspect of a wall 18 feet high is due south, and the length of the shadow cast on the north side at noon is 16 feet. Find the sun's altitude, or the angle of elevation of the sun above the horizon.

9. The angle which the earth's radius subtends at the sun being $8''{\cdot}57$; find the distance of the sun from the earth in terms of the earth's radius.

10. At a distance of 200 yards from the foot of a church tower, the angle of elevation of the top of the tower was 30°, and of the top of the spire on the tower 32°. Find the height of the tower and of the spire.

11. Two men are surveying: when each is at a distance of 200 yards from the flag-staff, the one finds the angle subtended by the position of his companion and the staff to be 30° 15′. Find how far they are apart.

12. In order to ascertain the height of a castle on the top of a cliff, I measured from my position 240 yards directly from the castle, and at the ends of this line, found the angles of elevation of the top of the castle to be 29° and 13° 16′; also, at the further end of the line, the castle's height required subtended an angle of 5° 15′.

13. Two ships, half a mile apart, find that the angles subtended by the other ship and a fort, are respectively 56° 19′ and 63° 41′. Find the distance of each ship from the fort.

14. From the summit of a tower, whose height is 108 feet, the angles of depression of the top and bottom of a vertical column, standing in the horizontal plane, are found to be 30° and 60° respectively. Required the height of the column.

15. From the top of a house 42 feet high, I found the angle of elevation of the top of a neighbouring steeple on the same horizontal plane, to be 14° 13′, and at the bottom of the house it was 23° 19′. Find the height of the steeple.

16. In walking towards a certain object, I found the angle of elevation of its top to be $2^\circ\ 19'\ 13''$ at one milestone, and after proceeding to the next milestone, I found the angle of elevation to be $3^\circ\ 28'\ 49''$. How much further should I have to walk before I reached it, the milestones and object resting on the same level?

17. Wishing to ascertain the height of a house standing on the summit of a hill of uniform slope, I descended the hill for 40 feet, and then found the height subtended an angle of $34^\circ\ 18'\ 19''$. On descending a further distance of 60 feet, I found this angle to become $19^\circ\ 14'\ 52''$. Find the height of the house.

18. Wanting to know the height of a castle on a rock, I measured a base line of 100 yards, and at one extremity found the angle of elevation of the castle's top to be $45^\circ\ 15'$, and the angle subtended by the castle's height to be $34^\circ\ 30'$; also the angle subtended by the top of the castle and the other extremity of the base line to be $73^\circ\ 14'$. At the other extremity the angle between the first extremity and the top of the castle was $73^\circ\ 18'$. Find the height of the castle.

19. A person ascends 70 yards up a slope of 1 in $3\frac{1}{2}$ from the edge of a river, and observes the angle of depression of an object on the opposite shore to be $2\frac{1}{4}^\circ$. Find the breadth of the river.

20. An object 12 feet high standing on the top of a tower subtends an angle of $1^\circ\ 54'\ 10''$ at a station which is 250 feet from the base. Find the height of the tower.

21. Having measured a base line of 400 yards, whose upper end was 24 feet higher than the lower one, in the same vertical plane with the top of a hill, I found the angles of elevation of the top of the hill from the lower and upper ends of the base line to be $5^\circ\ 14'$ and $3^\circ\ 17'$ respectively. Find the height of the hill.

22. In order to measure the distance between two inaccessible objects C and D, I measured a base line AB of 500 yards, and at its extremities determined the following angles: $CAB = 94^\circ\ 13'$, $DAB = 62^\circ\ 20'$, $DBA = 84^\circ\ 58'$, and $CBA = 41^\circ\ 16'$. Find the distance between C and D.

23. Wanting to know my distance from an object P on the other side of a river, and having no instrument for observing angles, I measured a base line AB of 500 yards, and from A and B measured directly in a line, away from P, distances of 175 yards to C and D; I then found my distances from B and A to be respectively, $CB = 500$ and $AD = 650$ yards. Find PA and PB.

24. A lighthouse was observed from a ship to bear N. 34° E., and after the ship had sailed due south for 3 miles, the same lighthouse bore N. 23° E. Find the distance of the lighthouse from each position of the ship.

25. Two objects, A and B, were observed from a ship to be at the same instant in a line with a bearing N. 15° E. The ship then

sailed N.W. for 5 miles, when it was observed that A bore E., and B bore N.E. Find the distance between A and B.

26. A privateer is lying 10 miles S.W. of a harbour, and observes a merchantman leave it in the direction of S. 80° E., at the rate of 9 miles an hour. In what direction, and at what rate, must the privateer sail in order to come up with the merchantman in $1\frac{1}{2}$ hours?

27. From the top of the peak of Teneriffe, the dip of the horizon is found to be $1° 58' 10''$. If the radius of the earth be 4000 miles, what is the height of the mountain?

28. Having given that two points, each 10 feet above the earth's surface, cease to be visible from each other over still water at a distance of 8 miles; find the earth's diameter.

29. What is the dip of the horizon from the top of a mountain $1\frac{3}{4}$ miles high, the radius of the earth being 4000 miles?

30. From the top of a mountain $1\frac{1}{2}$ miles high, the dip of the horizon was found to be $1° 34' 30''$; find the earth's diameter.

31. In a town are three remarkable objects, A, B and C, known to be distant from each other as follows: $AB=426{\cdot}75$, $AC=610$, and $BC=538{\cdot}5$. From my position S, I observe that B lies beyond the line AC, and within the angle ASC: and I find the angle $ASC=23° 9'$, and $ASB=14° 16'$. Find the distance of S from A, B and C respectively.

32. Having removed to the other side of the town, so that B lies on the side of AC next to S, and still within the angle ASC; I observe the angles ASC, ASB to be $15° 14'$ and $14° 15'$. Find SA, SB, and SC.

33. Having again moved so as to have A and C in a line with S (A being the nearer), I find the angle ASB to be $18° 17'$. Find SA, SB, and SC.

34. Three points of land, A, B and C, are at known distances from each other, namely $AB=63$, $AC=44$, and $BC=76$. At a boat in the piece of water between them the angles subtended by AB and BC are observed to be $89° 15'$ and $130° 45'$ respectively. Find the distances of the boat from A, B and C.

35. Being on a river, and observing a column on the banks, I find the angle of elevation of its top to be 30°, and the angle subtended by its top and a small island down the river to be $47° 25'$. After sailing past the column to this island, a distance of 450 yards, I find the angle subtended by the top and my former position to be $18° 30'$. Find the height of the column.

36. On the opposite bank of a river to that on which I stood, is a tower 216 feet high. With a sextant I ascertained the angle subtended at my eye by the height of the tower to be $47° 56'$. Find my distance from the foot of the tower, supposing my eye to be 5 feet above the level of the tower's foot.

EXPANSIONS, SERIES, ETC.

Ex. 19. Expand in terms of the cosines of multiples of θ—

1. $\cos^6\theta$; $\cos^8\theta$.
2. $\cos^5\theta$; $\cos^7\theta$.
3. $\sin^4\theta$; $\sin^8\theta$.
4. $\sin^6\theta$; $\sin^{10}\theta$.

Expand in terms of the sines of multiples of θ—

5. $\sin^5\theta$; $\sin^9\theta$.
6. $\sin^7\theta$; $\sin^{11}\theta$.

Expand in terms of the powers of $\sin\theta$ and $\cos\theta$—

7. $\sin 4\theta$; $\sin 7\theta$.
8. $\cos 5\theta$; $\cos 6\theta$; $\cos 7\theta$.
9. Expand $\tan 3\theta$; $\tan 8\theta$; in terms of the powers of $\tan\theta$.

By the exponential expressions for $\sin\theta$, $\cos\theta$; prove that—

10. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
11. $\cos 2\theta = \cos^2\theta - \sin^2\theta$.
12. $\sin\theta = (1-\cos\theta)\cot\dfrac{\theta}{2}$.
13. $\sin 2\theta = 2\sin\theta\cos\theta$.
14. $2\cos^2\theta = 1+\cos 2\theta$.
15. $\tan 2\theta = \dfrac{\sin\theta+\sin 3\theta}{\cos\theta+\cos 3\theta}$.

Ex. 20. Sum the series—

1. $1+\cos x+\cos 2x+\cos 3x+$ &c. to n terms.
2. $1+x\cos\theta+x^2\cos 2\theta+x^3\cos 3\theta+$ &c. to n terms.
3. $\sin\theta+\sin 2\theta+\sin 3\theta+$ &c. to n terms.
4. $\cos\theta+\cos 3\theta+\cos 5\theta+$ &c. in infin.
5. $\tan\theta+2\tan 2\theta+2^2\tan 4\theta+$ &c. to n terms.
6. $\operatorname{cosec}\theta+\operatorname{cosec} 2\theta+\operatorname{cosec} 4\theta+$ &c. to n terms.
7. $1+\cos\theta\cos\phi+\cos^2\theta\cos 2\phi+\cos^3\theta\cos 3\phi+$ &c. in infin.
8. $\cos\theta+\cos(\theta+\phi)+\cos(\theta+2\phi)+\cos(\theta+3\phi)+$ &c. to $n+1$ terms.
9. $\cos^2\theta+\cos^2(\theta+\alpha)+\cos^2(\theta+2\alpha)+\cos^2(\theta+3\alpha)+$ &c. to n terms.
10. $\tan\theta+\dfrac{1}{2}\tan\dfrac{\theta}{2}+\dfrac{1}{2^2}\tan\dfrac{\theta}{2^2}+\dfrac{1}{2^3}\tan\dfrac{\theta}{2^3}+$ &c. to n terms.
11. $1+\dfrac{x\cos\theta}{1}+\dfrac{x^2\cos 2\theta}{1.2}+\dfrac{x^3\cos 3\theta}{1.2.3}+$ &c. in infin.
12. $\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\frac{1}{4}\sin 4\theta+$ &c. in infin.
13. $x\sin\theta+\dfrac{x^2\sin 2\theta}{1.2}+\dfrac{x^3\sin 3\theta}{1.2.3}+$ &c. in infin.

Ex. 20. Sum the series—

14. $x \sin\theta - \frac{1}{2}x^2 \sin 2\theta + \frac{1}{3}x^3 \sin 3\theta - \frac{1}{4}x^4 \sin 4\theta +$ &c. in infin.

15. $(\cos\theta + \sqrt{-1}\sin\theta) + (\cos\theta + \sqrt{-1}\sin\theta)^2$
$+ (\cos\theta + \sqrt{-1}\sin\theta)^3 +$ &c. to n terms.

16. $(3^{\frac{1}{2}} - 1)3^{-\frac{1}{2}} - \frac{1}{3}(3^{\frac{3}{2}} - 1)3^{-\frac{3}{2}} + \frac{1}{5}(3^{\frac{5}{2}} - 1)3^{-\frac{5}{2}} - \ldots$ in inf.

17. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ in inf.

18. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ in inf.

19. Prove that, $\tan n\theta = \dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \text{\&c. to } n \text{ terms}}{\cos\theta + \cos 3\theta + \cos 5\theta + \text{\&c. to } n \text{ terms}}$.

20. If $\sin x = n \sin(a + x)$; express x in a series of sines of multiples of a.

21. If $\tan\theta(1 + m\sin\varphi) = m\cos\varphi$; express θ in a series of sines and cosines of multiples of φ.

In any triangle ABC prove that—

22. $\log_e\frac{a}{b} = (\cos 2B - \cos 2A) + \frac{1}{2}(\cos 4B - \cos 4A) + \frac{1}{3}(\cos 6B - \cos 6A) +$ &c.

23. $\log_e\frac{a}{c} = \frac{b}{a}\cos C + \frac{b^2}{2a^2}\cos 2C + \frac{b^3}{3a^3}\cos 3C +$ &c.

24. $\angle B = \frac{b}{c}\sin A + \frac{b^2}{2c^2}\sin 2A + \frac{b^3}{3c^3}\sin 3A +$ &c.

Ex. 21. Prove that—

1. $(a \pm b\sqrt{-1})^{\frac{1}{n}} = (a^2 + b^2)^{\frac{1}{2n}}\left(\cos\frac{\theta}{n} \pm \sqrt{-1}\sin\frac{\theta}{n}\right)$, if $\tan\theta = \frac{b}{a}$.

2. $\log_e \sec\theta = \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta -$ &c.; thence sum,
$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +$ &c. *in infinitum*.

3. $\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\sin\frac{\pi+\theta}{n}\sin\frac{2\pi+\theta}{n}\ldots\ldots\sin\frac{(n-1)\pi+\theta}{n}$.

4. $\tan\frac{\pi}{4n}\tan\frac{5\pi}{4n}\tan\frac{9\pi}{4n}\ldots\tan\frac{(4n-3)\pi}{4n} = \pm 1$;
according as n is of the form $4m + 1$, or $4m + 3$ respectively.

5. $\sin\theta\sin 3\theta\sin 5\theta\ldots\sin(2n+1)\theta = 2^{-n}$;
if $\frac{\pi}{2} = (2n+1)\theta$, and n be greater than $\frac{1}{2}$.

6. $\alpha = \tan\alpha - \frac{1}{3}\tan^3\alpha + \frac{1}{5}\tan^5\alpha - \ldots$, if α be $< \frac{1}{4}\pi$.

Ex. 21. Prove that—

7. $e^x+e^{-x}=2\left(1+\frac{2^2x^2}{\pi^2}\right)\left(1+\frac{2^2x^2}{3^2\pi^2}\right)\cdots$

8. $e^x-e^{-x}=2x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\cdots$

9. Find the sum of the straight lines drawn, from any one of the angular points of a regular polygon of n sides, each side of which is $2a$, to all the other angular points.

10. Find the product of all the lines that can be drawn from one of the angles of a regular polygon of n sides, inscribed in a circle of radius a to the other angles.

Ex. 22. Solve by Trigonometry the equations—

1. $x^3-6x=4$.
2. $x^3-147x-343=0$.
3. $x^3-3x+1=0$.
4. $x^3-3x-2=0$.
5. $x^4+1=0$.
6. $x^6-x^3+1=0$.

Ex. 23. Adapt to logarithmic computation—

1. $x=a^2+b^2$.
2. $x=(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}$.
3. $a\sin x+b\cos x=c$.
4. $x=(a^2+b^2-2ab\cos C)^{\frac{1}{2}}$.
5. $\cos x=\cos c\sin A\sin B-\cos A\cos B$.

APPLICATION OF ALGEBRA TO GEOMETRY.

1. Divide a straight line, one foot long, in extreme and mean ratio.

2. In a right-angled triangle, the base $=20$, and the difference between the hypothenuse and perpendicular $=8$; determine the triangle.

3. Given the sum of the base and perpendicular of a right-angled triangle $=49$, and the sum of its base and hypothenuse $=63$; determine the sides.

4. The area of a right-angled triangle being 54, and the hypothenuse 15; determine the sides.

5. Determine the right-angled triangle, in which the hypothenuse is 17, and the radius of the inscribed circle 3.

6. The area of a right-angled triangle $=840$, and the radius of the circumscribed circle $=29$; determine the sides.

7. The perimeter of a right-angled triangle is 20, and the radius of the inscribed circle $1\frac{1}{4}$; find the sides.

8. The perimeter of a right-angled triangle is 24, and the perpendicular from the right angle on the hypothenuse is $4\frac{4}{5}$; find the sides.

9. Given the hypothenuse of a right-angled triangle, and the side of an inscribed square: find the two sides of the triangle (1) when the given side coincides with the hypothenuse, and (2) when an angle of the square coincides with the right angle of the triangle.

10. The area of a right-angled triangle being $\frac{1}{2}h$, and the radius of the inscribed circle r; determine the sides.

11. CD is a perpendicular on the hypothenuse AB of a right-angled triangle; if r be the radius of the circle inscribed in ABC, and r_1, r_2 of those in CBD, ACD; show that $r^2=r_1^2+r_2^2$.

12. If the base of a triangle be 6, and the two sides 3 and 4; find the segments of the base made by a line bisecting the vertical angle.

13. Two sides of a triangle are 5 and 6·4, and the length of a line bisecting the vertical angle and meeting the base is 4; find the base.

14. The base of a triangle is 14, the difference of the two sides $3\frac{1}{2}$, and a perpendicular from the vertical angle on the base 8; determine the sides.

15. The base of a triangle is 34, the sum of the two sides 50, and a perpendicular from the vertical angle upon the base 8; determine the sides.

16. Given the base of a triangle $=50$, the altitude $=24$, and the radius of the inscribed circle $=10$; determine the sides.

17. Determine a triangle, having given $2d$ the sum of the two sides, p the perpendicular and $2n$ the difference of the segments of the base made by the perpendicular.

18. Given the base a, and the altitude p of a triangle; find the side of the inscribed square.

19. To find a triangle, such that its sides and a perpendicular on one of them from the opposite angle, may be in continued geometrical progression.

20. Given the three perpendiculars from the angles of a triangle upon the opposite sides; find the area and sides of the triangle.

21. Determine the sides of a triangle, which are in Ar. Prog. with the common difference $=1$, and in which the radius of the inscribed circle $=4$.

22. The sides of a triangle are in Ar. Prog., a, c being the longest and shortest sides: if R, r be the radii of the circumscribed and inscribed circles, show that $6Rr=ac$.

23. Given a the base of a triangle, $n:1$ the ratio of the two sides, and d the distance of the vertex from a given point in the base; determine the sides.

24. Given $2a$ the base, p the perpendicular and m^2 the rectangle of the two sides of a triangle; find the sides.

25. From the obtuse angle A of a given triangle, draw to the base a line, the square on which shall be equal to the rectangle of the segments of the base.

26. Draw a straight line from one angle of a square whose side is 30, so that the part intercepted, between one of the sides containing the opposite angle and the other side produced, shall equal 16.

27. If an isosceles triangle be inscribed in a circle having each of the sides double of the base, show that $15(\text{rad.})^2=(2\text{ side})^2$.

28. If a, b, c be the sides of a triangle, R, r the radii of the circumscribed and inscribed circles; show that

$$\text{the area}=\frac{abc}{4R}\text{ or }=\frac{(a+b+c)r}{2}.$$

29. If in the triangle ABC, the lines bisecting the angles ABC and meeting the opposite sides a, b, c be h, k, l respectively; prove

$$\text{that }\frac{1}{hkl}=\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\times\frac{\text{rad. of circums. circle}}{a+b+c}.$$

30. Upon the sides of a triangle ABC right-angled at C, having described semicircles towards the same parts AEB, ADC, BDC; show that the difference of the figures ADBE, CD is equal to the triangle ABC.

31. The centres of three circles (A, B, C) are in the same straight line, B and C touch A internally, and each other externally; show that the portion of A's area which is outside B and C is equal to

the area of the semicircle described on the chord of A which touches B and C at their point of contact.

32. Through a point M equidistant from two straight lines AA′, BB′ at right angles to each other, to draw a straight line PMQ, so that the sum of the squares upon PM and MQ shall be equal to the square upon a given line b.

33. To inscribe a semicircle in a quadrant.

34. Find the side of a square, and the radius of a circle inscribed in a given quadrant.

35. To inscribe a circle in a given sector of a circle.

36. If a, b, c be the chords of three adjacent arcs of a circle whose sum equals the semicircumference, of which x is the radius; prove that $4x^3-(a^2+b^2+c^2)x-abc=0$.

37. Given the chords of two arcs of a given circle; find the chord of their sum, and the chord of their difference.

38. Given the lengths of two chords of a circle which intersect at right angles, and the distance of their point of intersection from the centre; find the diameter of the circle.

39. If a circle be described, so as to touch the side a of a triangle externally, and the sides b, c, produced; find its radius.

40. The radii of two circles which intersect one another are r, r'; and the distance of their centres is c; find the length of their common chord.

41. In a given square to inscribe another square having its side equal to a given straight line. What are the limits of this line?

42. If from one of the angles of a rectangle, a perpendicular be drawn to its diagonal d, and from the point of their intersection, lines p, q be drawn perpendicular to the sides which contain the opposite angle; show that, $p^{\frac{2}{3}}+q^{\frac{2}{3}}=d^{\frac{2}{3}}$.

43. If the distances of any given point within a square to three of its angular points be h, k, l; determine a side of the square.

44. If h, k, l be the sides of a regular pentagon, hexagon and decagon respectively, inscribed in the same circle; show that

$$h^2=k^2+l^2.$$

45. The radius of a circle being 1, find the areas of the inscribed and circumscribed equilateral triangles; hence find the area of the inscribed regular hexagon.

46. Show that the area of a dodecagon inscribed in a circle is equal to that of a square on the side of an equilateral triangle inscribed in the same circle.

47. Find the radius of the circle inscribed in a given rhombus.

48. Show that, if in a quadrilateral figure the sums of the opposite sides are equal, a circle may be inscribed in it.

49. A circle is inscribed in an equilateral triangle, an equilateral triangle in the circle, a circle again in the latter triangle, and so on; if $r, r_1, r_2, r_3, \ldots$ be the radii of the circles, prove that $r=r_1+r_2+r_3+\ldots$

50. If r be the radius of the circle inscribed between the base of a right-angled triangle and the other two sides produced, and r' be the radius of the inscribed circle between the altitude of the same triangle and the other two sides produced; prove that

$$\text{the area of the triangle} = rr'.$$

51. In a given square a circle is inscribed; in one of the interstices between this circle and square another circle is inscribed; and in the interstice between the second circle and square a third is inscribed, and so on to infinity; find the sum of the circular areas.

52. If O be the centre of the circle (radius r) inscribed in a triangle ABC; OL, OM, ON perpendiculars on the sides a, b, c; r_1, r_2, r_3 the radii of the circles inscribed in the quadrilaterals AO, BO, CO; prove that $\dfrac{r_1}{r-r_1}+\dfrac{r_2}{r-r_2}+\dfrac{r_3}{r-r_3}=\dfrac{a+b+c}{2r}$.

53. If R, r be the radii of two spheres inscribed in a cone, so that the greater (R) may touch the less, and also the base of cone; find the volume of the cone.

54. Into a hollow cone, whose vertical angle is 2α, are put a number of spheres one above another, each sphere touching the one above, the one below, and the sheet of the cone; show that the radii of the spheres are in Geometrical progression.

55. Divide the frustum of a cone into three equal parts, by two planes parallel to the base.

56. Find the solid content of a triangular pyramid whose faces are four equal isosceles triangles, of which the sides are 12, 12 and 10.

ANALYTICAL GEOMETRY

AND

CONIC SECTIONS.

I. Straight Line.

Ex. 1. Draw the straight lines whose equations are—

1. $y=3x+1$.
2. $y-3=5x+2$.
3. $5y-4x=0$.
4. $1-2x+3y=0$.
5. $\frac{x}{3}+\frac{y}{7}=2$.
6. $2x+3=0$.

7. Find the equation to a straight line which passes through the two points $x=2, y=5$; $x=0, y=-7$.

8. Find the equation to a straight line which lies evenly between two given parallel straight lines.

9. Determine the point of intersection of two lines whose equations are $3y-x=0$, $2x+y=1$.

10. Find the equation to a straight line which passes through the point $x=1$, $y=3$; and makes an angle of $30°$ with the straight line whose equation is $2y-x+1=0$.

11. Find the equation to a straight line which passes through a given point in the axis of x at a distance c from the origin and makes an angle of $45°$ with the straight line whose equation is $\frac{x}{a}-\frac{y}{b}=1$.

12. Find the equation to a straight line which is perpendicular to the line $4y+5x-3=0$, and cuts the axis of y at a distance $=7$ from the origin.

13. Find the distance of the origin of coordinates from the line whose equation is $\frac{1}{5}x+\frac{1}{2}y=1$.

14. Find the distance of the point of intersection of the lines $2x-3y+5=0$, $3x+4y=0$, from the line $y=2x+1$.

15. Find the cosine of the angle contained between two straight lines whose equations are $y-3x+c=0$, and $y-5x+d=0$.

16. Find the angle included between the lines whose equations are $1+3x+2y=0$, $3+2x-3y=0$.

17. Find the tangent of the angle contained by the lines $2x+3y+4=0$, $3x+4y+5=0$.

18. Find the equation to a straight line which bisects the angle included between the lines $5y-2x=0$, $3y+4x=12$.

Ex. 1.

19. The equation $(2y+x)(3y-x)=0$, represents two straight lines inclined to one another at an angle of $135°$.

20. The equation $2y^2-3xy-2x^2-y+2x=0$, represents two straight lines at right angles to one another.

21. The equation $y^2-xy-2x^2+5x-y-2=0$, represents two straight lines inclined to each other at an angle $\tan^{-1}3$.

22. The equation $y^2-2xy\sec\theta+x^2=0$, represents two straight lines that include an angle $=\theta$.

23. Find the length of a perpendicular drawn from the point $x=3$, $y=5$, upon the line $7x-3y=9$.

24. Find the length of the perpendicular from the vertex upon the base of a triangle; the coordinates of the vertex being $(3, 5)$, and those of the extremities of the base $(1, 3)$ and $(2, 0)$.

25. Determine the geometrical signification of the equations—

(1) $12xy+8x-27y-18=0$.
(2) $x^2+y^2+xy=0$.
(3) $y^2-2xy+2x^2-2x+1=0$.
(4) $y^2+4xy+x^2-2y+2x-2=0$.
(5) $y^2-2xy+3x^2-2y-10x+19=0$.
(6) $y^2-2xy+x^2-2y+2x=0$.
(7) $\frac{5}{4}x^2-xy+y^2-2x+1=0$.
(8) $y^3-9c^2y=0$.

26. What is the inclination of the coordinate axes to one another, when the two lines represented by the equation $y^2-x^2=0$, are perpendicular to one another?

27. The equations of two straight lines, referred to oblique axes inclined at an angle ω, being $y-mx=0$, $my+x=0$; find the angle between them.

28. If $\dfrac{x}{a+b\cos\omega}+\dfrac{y}{b+a\cos\omega}=1$, be the equation to a straight line referred to axes inclined to one another at an angle ω; find the equation to, and the length of, a perpendicular dropped from a point $(a, b,)$ upon this line.

29. Find the polar equation to a straight line; and trace the line whose equation is $r=2a\cos\left(\theta+\dfrac{\pi}{6}\right)$.

30. The polar equation to a straight line, which passes through the two points, whose polar coordinates are (r_1, α_1), (r_2, α_2) is

$$\frac{\sin(\alpha_2-\alpha_1)}{r}+\frac{\sin(\theta-\alpha_2)}{r_1}+\frac{\sin(\alpha_1-\theta)}{r_2}=0.$$

31. In the equation $ay^2+bxy+dy+ex=0$, find the relation among the coefficients, that it may represent two straight lines.

32. Find the area included between the lines,

$$x=0, \qquad y=0, \qquad 5x+4y=20.$$

Ex. 1.

33. Find the area included between the lines,

$$y=c, \qquad y=3x, \qquad y=7x.$$

34. Find the area of the triangle included between the lines,

$$x+y=1, \qquad x-y=0, \qquad y=0.$$

35. Find the area of the triangle included between the lines,

$$x+2y=5, \qquad 2x+y=7, \qquad y=x+1.$$

36. Find the area included between the lines,

$$y=x\tan\alpha, \qquad y=x\tan\alpha_1, \qquad y=x\tan\alpha_2+a.$$

37. Find the area of the triangle included between the lines,

$$y=ax+b, \qquad y=a'x+b', \qquad y=a''x+b''.$$

38. Find the area included between the lines,

$$x-y=0, \qquad x+y=0, \qquad x-y=a, \qquad x+y=b.$$

39. Express the area of a triangle in terms of the coordinates of its angular points a, b; a', b'; a'', b''.

40. Show that the lines, of which the equations are,

$$y=2x+3, \qquad y=3x+4, \qquad y=4x+5,$$

all pass through one point.

II. Circle.

If h, k *be the rectangular coordinates of the centre of a circle,* c *its radius; then* $(x-h)^2+(y-k)^2=c^2$, *is the equation to the circle.*

Ex. 2. Find the centre and radius of the circle—

1. $x^2+y^2+4x-6y-3=0.$
2. $x^2+y^2+12x-8y+48=0.$
3. $4x^2+4y^2-8x+16y+19=0.$
4. $x^2+y^2-2cx+6cy+9c^2=0.$
5. $(x+y-a)^2=xy$; the inclination of the axes being $\frac{1}{3}\pi$.
6. $r^2-2(\cos\theta+\sqrt{3}\sin\theta)r-5=0.$
7. Find the equation to the straight line, which passes through the centres of the circles, $x^2+4x+y^2+6y=3$; $x^2+y^2+2y=0$.
8. Find the equation to a circle which passes through the three points, $(1, 2)$, $(1, 3)$, $(2, 5)$.
9. Find the equation to a circle, having for diameter the distance of a given point from the origin of coordinates.
10. Find the equation to a line touching a given circle, and parallel to a given straight line.
11. Find the equation to the straight line, which passing through the origin, is a tangent to the circle, $x^2+y^2-3x+4y=0$.
12. Find the equation to tangents to the circle $x^2+y^2=9$, at the points whose common abscissa is $x=2$.
13. Find the equation to a circle whose centre is at the origin, and of which the line $y=3x+2$ is a tangent.

Ex. 2.

14. If the line $\frac{x}{a}+\frac{y}{b}=1$, be a tangent to the circle $x^2+y^2=c^2$; find the relation between a, b, c.

15. A straight line is drawn from a point (h, k) so as to touch the circle $x^2+y^2=c^2$, and to cut off a portion p from the axis of y; find an equation for determining p in terms of h, k and c.

16. Find the equation to a circle, which touches a given straight line, and passes through two given points.

17. Find the equation to a circle, of which the radius is c, and which is referred to two rectangular tangents as axes.

18. Find the equation to a circle, the diameter of which is the common chord of the two circles $x^2+y^2=c^2$, $(x-a)^2+y^2=c^2$.

19. Find the equation to the common chord of the two circles,

$$x^2-4x+y^2-2y-11=0; \qquad x^2+6x+y^2+4y-3=0.$$

20. If $\frac{x}{a}+\frac{y}{b}=1$, be the equation to a chord of the circle $x^2+y^2=c^2$; find its length.

21. If $y=mx$, be the equation to a chord of the circle $x^2+y^2=2rx$; show that the equation to the circle of which that chord is a diameter is $(1+m^2)(x^2+y^2)-2r(x+my)=0$.

22. Find the equation to the chord of a given circle, which subtends a right angle at the centre, and of which the position of one extremity is given.

23. If, on any three chords, drawn from the same point in the circumference of a circle, as diameters, circles be described; the points of intersection of these circles lie in one straight line.

24. If AB be the diameter of a circle, MN a chord parallel to AB, P any point in AB; show that $PM^2+PN^2=AP^2+PB^2$.

25. From any point Q in the circumference of a semicircle, two chords are drawn to the extremities A, B of its diameter. If, from any point N in this diameter, a perpendicular be drawn, cutting AQ in H, BQ in K, and the curve in P; prove that

$$NH.NK=NP^2.$$

26. If 2α, 2β be the inclination to the axis of x of two radii of the circle, $x^2+y^2=c^2$; find the equation to the chord joining their extremities.

27. Find the radius of a circle inscribed in the triangle, of which the equations to the sides are,

$$x\cos\alpha+y\sin\alpha=a, \quad x\cos\beta+y\sin\beta=b, \quad x\cos\gamma+y\sin\gamma=c.$$

28. Find the equation to a circle referred to oblique axes.

29. If $x^2-xy+y^2-ax-ay=0$ represent a circle; find the inclination of the coordinate axes, and the radius.

30. Find the diameter of the circle represented by the equation

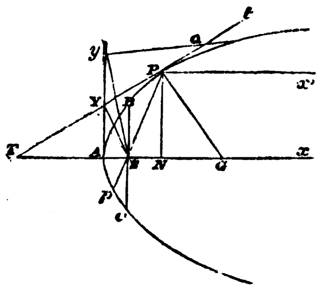
$$x^2+2xy\cos\omega+y^2=bx+cy.$$

III. Parabola.

General properties of the Parabola.

If AS=a, AN=x, NP=y, S *being the focus*—

1. $y^2=4ax$, *the rectangular equation.*
2. BC *the latus rectum* $=4a$.
3. $SP=a+x$.
4. $yy'=2a(x+x')$ *the equation to* Tt, $x'y'$ *being the coordinates of* P.
5. TN *the subtangent* $=2x$.
6. NG *the subnormal* $=2a$.
7. $SY^2=ST\times SA=SP\times SA$.
8. $\angle SPT=\angle tPx'$.
9. *Area* $APN=\frac{2}{3}AN\times PN$.
10. $r=\dfrac{2a}{1+\cos\theta}$, *if* $SP=r$, $\angle ASP=\theta$; *the polar equation.*



Ex. 3. Find the position and dimensions of the curves—

1. $1+2x+3y^2=0$.
2. $y-\frac{1}{3}x^2=2$.
3. $y^2-2xy+x^2-8x+16=0$.
4. $y^2+2xy+x^2+y-3x+1=0$.
5. $y^2-2xy+x^2-6y-6x+9=0$.
6. $y^2+4xy+4x^2+3ax+a^2=0$.
7. $(y+c)^{\frac{1}{2}}+(x+c)^{\frac{1}{2}}=2c^{\frac{1}{2}}$.

8. The rectangle contained between two ordinates y_1, y_2 of the parabola $y^2=4ax$, is equal to a^2, the distance between them being a; find the values of y_1, y_2.

9. Two tangents to a parabola drawn from the same point of the directrix are at right angles to each other.

10. The tangent at any point of a parabola will meet the directrix and latus rectum produced, in two points equally distant from the focus.

11. If two equal parabolas have a common axis, a straight line touching the interior parabola, and bounded by the exterior, will be bisected by the point of contact.

12. Describe a parabola which shall touch a given circle at a given point, and have its axis coincident with a given diameter of the circle.

13. If a parabola intersect a circle in four points; prove that the sums of the ordinates of the points of intersection on opposite sides of the axis are equal to each other.

14. Find the ordinate of a point in a parabola such, that a tangent being drawn to the curve at this point, the intercepts on the axes of coordinates may be equal to one another.

15. The abscissa and double ordinate of a parabola are h and k;

Ex. 3.

the diameters of its circumscribed and inscribed circles are D and d; prove that $D+d=h+k$.

In solving this question, we find that $D=4a+h$ *and* $d=-4a+k$.

16. A straight line inclined at an angle θ to the axis of x touches both the curves, $y^2=4ax$, $x^2+y^2=c^2$; find the value of θ.

17. In a parabola $y^2=lx$, the ordinates of three points, such that the normals pass through the same point, are y_1, y_2, y_3; show that $y_1+y_2+y_3=0$, and find the equation to the circle passing through these three points.

18. If from the point of contact of a tangent to a parabola, a chord be drawn and a line parallel to the axis meeting the chord, the curve and the tangent; show that this line will be divided by them in the same ratio as it divides the chord.

19. Find the equation to the circle of curvature at the extremity of the latus rectum of a parabola.

20. Two straight lines which are always tangents to a given parabola, are so inclined to the axis of x that the sum of the cotangents of the angles which they make with that axis is constant; prove that the locus of their intersections is a straight line parallel to the axis.

21. Find the locus of the intersection of two straight lines, which always touch a parabola, the product of the cotangents of their inclinations to the axis being a constant quantity c.

22. Two tangents to a parabola make angles θ, θ' with its axis; find the locus of their intersection, (1) when $\sin\theta\sin\theta'=m$; (2) when $\cot\theta-\cot\theta'=n$.

23. From the vertex of a parabola a straight line is drawn, inclined at 45° to the tangent at any point; find the equation to the curve which is the locus of their intersections.

24. If PT, QT be two tangents at the points P, Q of a parabola whose focus is S, then $SP.SQ=ST^2$; and if SP, SQ include an angle α, the locus of T will be a hyperbola whose eccentricity $=\sec\frac{1}{2}\alpha$.

25. PT, QT are two tangents to a parabola; any other tangent cuts these two in points p, q respectively; prove that $\frac{Tp}{TP}+\frac{Tq}{TQ}=1$.

26. PT, QT are two equal tangents to a parabola, P, Q being the points of contact; if a third tangent cut PT in E, and QT in F; prove that PE = FT, and QF = ET.

27. If SY be the perpendicular drawn from the focus of a given parabola upon the tangent at any point P; find the locus of the centre of the circle circumscribing the triangle SPY.

28. Two parabolas have a common focus and axis, and a tangent to the one parabola intersects a tangent to the other at right angles; find the locus of the point of intersection.

Ex. 3.

29. If, from the focus of a parabola, lines be drawn to meet the tangents at a constant angle; prove that the locus of the points of intersection is that tangent to the parabola the inclination of which to the axis is equal to the given angle.

30. Draw a normal at the extremity of the latus rectum of a parabola whose equation is $y^2=4a(x-a)$, and find its distance from the origin of coordinates.

31. If PQ be the chord of a parabola, which is a normal at P, and the tangents at P and Q intersect in a point T; prove that the line PT is bisected by the directrix.

32. Find the equation to the normal of a parabola, which is inclined at a given angle to the axis of the curve.

33. Find the locus of the intersection of a straight line drawn from the focus of a parabola, perpendicular to the normal at any point.

34. The locus of the intersection of two normals to a parabola at right angles to one another, is a parabola whose latus rectum is one-fourth of the latus rectum of the original.

35. If several parabolas have the same vertex and axis, the locus of the extremities of normals to them from a given point in the axis is an ellipse.

36. The area between two normals to a parabola at the extremities of a focal chord, and the curve, is equal to $20a^2 \div 3 \sin^3 2\theta$, θ being the inclination of one of the normals to the axis.

37. Find the locus of the intersections of the normals at any two points of a parabola, on opposite sides of its axis, the ordinates of which are as 1 to 2.

38. Find the area of the triangle included between the tangents to the parabolæ, $y^2=4ax$, $y^2=4mx$, at points, the common abscissa of which is h, and the portion of the ordinate intercepted between the two curves.

39. Having given b, c the lengths of two tangents to a parabola at right angles to one another; show that the latus rectum $=4b^2c^2 \div (b^2+c^2)^{\frac{3}{2}}$.

40. If one side of a triangle and two others produced be tangents to a parabola, and the points of contact be joined, a triangle will be formed whose area is double of that of the exterior triangle.

41. Two tangents h, k, to a parabola, intersect one another at an angle $=\omega$, and a circle is described between the tangents and curve; show that the diameter $=hk \sin \omega \div (h+k+2\sqrt{hk} \, . \sin \frac{1}{2}\omega)$.

42. A triangle is formed by the meeting of three tangents to a parabola. Show that the products of the alternate segments of the tangents made by their mutual intersections are equal.

43. A circle described through the intersections of three tangents of a parabola will pass through the focus.

Ex. 3.

44. If P be any point of a parabola whose vertex is A, and QSQ′ be the focal chord parallel to AP; PN, QM, Q′M′ being perpendicular to the axis; show that $SM^2 = AM.AN$, and $MM' = AP$.

45. If a circle, described upon a chord of a parabola as diameter, meet the directrix, prove that it also touches it; and show that all the chords for which this is possible intersect in the same point.

46. If PSp be any focal chord of a parabola whose vertex is A; prove that AP, Ap will meet the latus rectum in two points Q, q, of which the distances from the focus are equal to the ordinates of p and P respectively.

47. If chords be drawn to a parabola, all passing through the point in which the axis and directrix of the curve intersect; find the locus of their middle points.

48. If PSp be a focal chord of a parabola, RDr the directrix meeting the axis in D, Q any point in the curve; if PQ, pQ produced meet the directrix in R, r; prove that $DR \times Dr = (2a)^2$.

49. Find the locus of the intersection of a tangent at one extremity of a focal chord with the ordinate at the other produced.

50. In the focal distance SP take Sp equal to the ordinate PN. Find the equation to the curve traced out by the point p.

51. If SL be drawn, from the focus S of a parabola perpendicular to the normal at any point P; show that the abscissa of L is equal to SP; and prove, AN being the abscissa of P, that $SL^2 = AN.SP$.

52. If PSp be any focal chord of a parabola; prove that

$$SP \times Sp = a(SP + Sp).$$

53. Given the radius vector at any point of a parabola and the angle it makes with the curve; find the latus rectum and the position of the vertex.

54. If QSq be a focal chord of a parabola, drawn parallel to the tangent at a point P; prove that $SQ \,.\, Sq = 4a \,.\, SP$.

55. The abscissæ of two points in a parabola, measured along the axis are h, $3h$; and the corresponding focal distances r, $2r$: show that $h = a$.

56. If r, r' be two radii vectores of a parabola, the focus being the pole, at right angles to each other, prove that

$$\left(\frac{1}{r} - \frac{1}{2a}\right)^2 + \left(\frac{1}{r'} - \frac{1}{2a}\right)^2 = \frac{1}{(2a)^2}.$$

57. If, from any point P in a parabola, a perpendicular PH be drawn to the directrix, prove that SP, SH, and the latus rectum are in geometrical progression.

58. If PSp be any focal chord of a parabola; prove that the rectilineal triangle $PAp \propto (Pp)^{\frac{3}{2}}$.

59. If, the vertex of a parabola being the pole, r_1, r_2 be two radii

Ex. 3.

vectores at right angles to each other; prove that

$$(4a)^2 = (r_1 r_2)^{\frac{4}{3}} \div (r_1^{\frac{2}{3}} + r_2^{\frac{2}{3}}).$$

60. In the axis of a parabola is taken a point E, at a distance from the vertex equal to $2a$; P is any point in the curve; join EP, PN is an ordinate to the axis: prove that $EP^2 - EA^2 = AN^2$.

61. If from any point D in the arc of a parabola, a straight line DE be drawn parallel to the axis of the curve, to meet a chord Pp produced if necessary, in E; prove that $EP . Ep \propto ED$.

62. If BC, CD be two consecutive arcs of a parabola, the sagittæ of which drawn parallel to the axis are equal; prove that the chord of BCD is parallel to the tangent at C.

Def. The sagitta of an arc is a straight line drawn from the middle point of the chord to meet the arc.

63. If AB, BC be two consecutive arcs of a parabola, and LW, GH, MV be the sagittæ, drawn parallel to the axis of the arcs AB, AC, BC respectively; prove that $\sqrt{LW} + \sqrt{MV} = \sqrt{GH}$.

64. Qq is a chord of a parabola: from any point R in this chord is drawn REF parallel to the axis of the parabola, meeting the curve in E, and the tangent drawn through the point Q in F; prove that $RQ : Rq = EF : ER$.

65. Find the latus rectum of the parabola $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$.

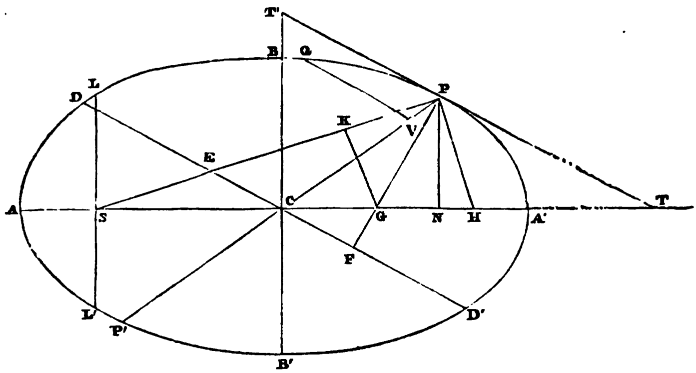
66. Find the area included between the parabola $y^2 = 4ax$, and the straight line $x = y + a$.

67. Find the locus of the centre of a circle inscribed in a sector of a given circle, one of the bounding radii of the sector remaining fixed.

68. In a plane triangle ABC, if $\tan A \tan \frac{1}{2}B = 2$, and AB be fixed; find the locus of C.

IV. Ellipse.

General Properties of the Ellipse.



If $CA=a$, $CB=b$, $\dfrac{CS}{CA}=e$; $CN=x$, $NP=y$;

1. $y^2=\dfrac{b^2}{a^2}(a^2-x^2)$, *the rectangular equation.*
2. LL' *the lat. rect.* $=\dfrac{2b^2}{a}$ *or* $2a(1-e^2)$.
3. $SP=a+ex$, $HP=a-ex$, S, H *being the foci.*
4. $a\,yy'+b^2xx'=a^2b^2$ *the equation to the tangent at* P.
5. $CT=\dfrac{a^2}{x}$, $CT'=\dfrac{b^2}{y}$; x, y *the coordinates of* P.
6. $CG=e^2x$, $GN=\dfrac{b^2}{a^2}x$, PG *being the normal at* P.
7. $QV^2=\dfrac{CD^2}{CP^2}\times PV\,.\,VP'$, *where* QV *and* CD *are parallel to* PT.
8. $CP^2+CD^2=a^2+b^2$, *and* $CD^2=SP.PH$.
9. $CD.PF=ab$.
10. *Area of ellipse* $=\pi ab$.
11. $r=\dfrac{a(1-e^2)}{1+e\cos\theta}$; *when* $SP=r$, $\angle ASP=\theta$.
12. $r=\dfrac{b}{(1-e^2\cos^2\theta)^{\frac{1}{2}}}$; *when* $CP=r$, $\angle ACP=\theta$.

(11, 12) *polar equations to the ellipse.*

13. *At* P, *the radius of curvature* $\rho=\dfrac{CD^3}{ab}=\dfrac{a^2}{b^4}.PG^3$.

Ex. 4. Determine the position and axes of the curves—

1. $3x^2+2y^2-2x+y-1=0$.

Ex. 4. Determine the position and axes of the curves—

2. $x^2+2y^2-2x+4y-6=0$.
3. $5x^2+2xy+5y^2-12x-12y=0$
4. $3x^2+2xy+3y^2-16y+23=0$.
5. $x^2+xy+y^2+x+y-5=0$.
6. $3x^2+2xy+y^2-4x=0$.
7. $5x^2+6xy+5y^2-26x-22y+29=0$.
8. $16x^2+16xy+7y^2+64x+32y+28=0$; the coordinates being inclined to each other at an angle $\frac{1}{3}\pi$.

9. If A, S, C be the vertex, focus and centre of an ellipse; it is required to show that if AC become infinite, AS remaining finite, the ellipse will be changed into a parabola.

10. In an ellipse, find the position of that focal distance SP which is a mean proportional to the semi-axes; when $a=50$, $b=30$.

11. If a circle be described touching the axis major of an ellipse in one of the foci, and passing through one extremity of the axis minor, the semi-major axis will be a mean proportional between the diameter of the circle and the semi-axis minor.

12. Find the eccentricity of the ellipse whose equation is $2x^2+3y^2=c^2$.

13. In any ellipse, show that $\tan\frac{\text{PSH}}{2}\tan\frac{\text{PHS}}{2}=\frac{1-e}{1+e}$.

14. If PSQ be a focal chord of an ellipse and X the foot of the directrix; show that XP, XQ are equally inclined to the axis.

15. P is any point in an ellipse whose foci are S and H; r, R are the radii of the circles inscribed in and circumscribed about the triangle SPH; prove that $\text{R}r \propto \text{SP.HP}$.

16. If from the extremities of the minor axis of an ellipse, two straight lines be drawn through any point in the curve and intersect the major axis in Q and R; prove that $\text{CQ}.\text{CR}=\text{CA}^2$.

17. If P be any point in an ellipse, AA′ its major axis, and PN the ordinate, and to any other point Q in the curve, AQ, A′Q be drawn, meeting NP in R and S; show that $\text{NR}.\text{NS}=\text{NP}^2$.

18. If CP be any semidiameter of an ellipse, and AQO be drawn from the extremity of the major axis parallel to CP, and meeting the curve in Q and the minor axis produced in O; show that $2\text{CP}^2=\text{AO}.\text{AQ}$.

19. If AR be drawn perpendicular to the major axis of an ellipse through its extremity A and equal to half the latus rectum; and an ordinate NP to the axis cut the straight line CR in Q; prove that $\text{NP}^2=2$ area QNAR.

20. Find the equation to the tangent to the ellipse $4x^2+9y^2=36$, at the extremity of the latus rectum.

21. A tangent at L (see figure) meets any ordinate NP produced in R; show that NR=SP.

22. A tangent at the extremity of the latus rectum intersects

Ex. 4.

the minor axis produced, in the circumference of a circle on the major axis.

23. If h, k represent the intercepts of the major and minor axes respectively made by any tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; prove that $\frac{a^2}{h^2}+\frac{b^2}{k^2}=1$.

24. The length of the perpendicular upon the tangent from the centre of an ellipse is equal to $a(1-e^2\cos^2\phi)^{\frac{1}{2}}$, where ϕ is the inclination of the tangent to the axis major.

25. If λ be the acute angle between the tangent and focal distance at any point of an ellipse, the distance of that point from the centre is equal to $(a^2-b^2\cot^2\lambda)^{\frac{1}{2}}$.

26. In an ellipse the tangents at the extremities of any focal chord intersect in the directrix and in a perpendicular to the chord from the focus.

27. Find the equation to a tangent to an ellipse $2x^2+y^2=2$, inclined at an angle of $45°$ to the axis of x.

28. The tangent to an ellipse is inclined at an angle θ to the major axis; show that the product of the distances of the extremities of the major axis from this tangent is equal to $b^2\cos^2\theta$.

29. If θ be the angle which the focal distance to any point of an ellipse makes with the tangent, and ϕ the angle between the lines drawn from that point to the extremity of the axis major, then $2\tan\theta=e\tan\phi$.

30. If a line be drawn through the centre of an ellipse, cutting the axis major at an angle θ, and the curve at an angle ϕ; prove that $(a^2-b^2)\cos(2\theta-\phi)=(a^2+b^2)\cos\phi$.

31. Find the area of the triangle contained between the axes produced, and a tangent inclined to the major axis at an angle θ.

32. If two tangents to an ellipse make angles θ, θ' with the major axis, such that $\tan\theta\tan\theta'$ is always equal to a constant c; find the locus of their intersection.

33. The tangent at a point P of an ellipse, of which C is the centre, meets the axes in T, t; if CP produced meet in L the circle described about the triangle TCt, show that PL is half the chord of curvature at P in the direction of C, and that the rectangle CP.CL is constant.

34. If l, m be the cosines of the angles which the normal, at the point $x'y'$ of an ellipse, makes with the axes; show that the equation to the tangent may be reduced to the form $lx+my=(l^2a^2+m^2b^2)^{\frac{1}{2}}$.

35. Find the equation to the normal at the extremity of the latus rectum in the ellipse $3x^2+4y^2=9$.

36. If from G the foot of the normal at P a perpendicular GK be drawn to either focal distance, then PK will equal half the latus rectum.

Ex. 4.

37. If PG, PG′ be parts of the normal cut off by the axes major and minor respectively, prove that $\frac{PG}{PG'}=\frac{b^2}{a^2}$.

38. Prove that GP (see figure) is the shortest line from G to the curve.

39. In any ellipse prove that $GK=e.PN$. (See figure.)

40. Find the equation to, and the length of, the normal at any point of an ellipse, in terms of its inclination to the axis major.

41. Prove that, $\tan CPG=\frac{e^2}{b^2}xy$; also, $=\frac{a-b}{a}\sin 2PGS$: see fig.

42. If from a point R in the tangent at the extremity B of the minor axis of an ellipse, a tangent RZ be drawn, prove that $RZ=$chord BZ, if $BR=\sqrt{3}.AC$.

43. Prove that the locus of the points of bisection of any number of chords of an ellipse which pass through the same point is an ellipse; and find the magnitude and position of the axes when the coordinates to the point are given.

44. P is a point in an ellipse, D a point in the major axis, such that PD is equal to the semi-minor axis; PQ is a normal at P meeting in Q a perpendicular to the major axis through D: find the locus of Q.

45. If perpendiculars drawn from the centre and focus of an ellipse (see figure) meet the tangent at P in Y, Z; prove that

$$\frac{TY}{PY}=\left(\frac{TZ}{PZ}\right)^2.$$

46. Find the locus of the centre of a circle inscribed in the triangle SPH.

47. If SP, HP be any two focal distances in an ellipse whose vertex is A; and if AQL be drawn cutting SP in Q and bisecting HP in L; show that the locus of Q is an ellipse whose axes are $2a(1-e)$ and $2b(1-e)$.

If S *be taken for the origin of coordinates,* x, y *the coordinates of* Q, x′y′ *those of* P, *then we may show that* $x'=\frac{2x}{1-e}$ *and* $y'=\frac{2y}{1-e}$.

48. If SY, HZ be perpendiculars from the foci upon the tangent at any point P of an ellipse, then SZ and HY will intersect in the middle point of the normal at P: and the locus of their intersection will be an ellipse whose axes are $a(1+e^2)$ and $a(1-e^2)^{\frac{1}{2}}$.

49. If a right line be drawn from the extremity of any diameter of an ellipse to the focus, the part intercepted by the conjugate diameter is equal to the semi-axis major.

50. CP and CD are conjugate semi-diameters of an ellipse;

Ex. 4.

normals at P and D intersect in R: prove that CR is perpendicular to PD.

51. Two conjugate diameters of an ellipse are produced to meet the same directrix, and from the point of intersection of each, a perpendicular is drawn to the other; these perpendiculars will intersect in the nearer focus.

52. If at the extremities of conjugate semi-diameters of an ellipse, normals be drawn; show that the sum of the squares of these normals $=a^2(1-e^2)(2-e^2)$, or $\frac{b^2}{a^2}(a^2+b^2)$.

53. If from the extremities of any diameter of an ellipse chords be drawn to any point in the curve, and one of them be parallel to a diameter, the other will be parallel to the conjugate diameter.

54. The diameters, which bisect the lines joining the extremities of the axes of an ellipse, are equal and conjugate.

55. If CP, CD be two semi-diameters of an ellipse at right angles to each other, prove that the distance of the centre from the chord PQ is equal to $ab \div (a^2+b^2)^{\frac{1}{2}}$.

56. In any ellipse, prove that PE=AC. (See figure.)

57. From any point P of an ellipse, a straight line PQ is drawn perpendicular to the focal distance SP, and meeting in Q the diameter conjugate to that through P; show that PQ varies inversely as the perpendicular from P on the major axis.

58. If two equal and similar ellipses have the same centre; show that their points of intersection are at the extremities of diameters at right angles to one another.

59. If in an ellipse the diameter conjugate to CP meet SP, and HP (or these produced) in E and E′; prove that SE is equal to HE′, and that the circles which circumscribe the triangles SCE, HCE′ are equal to one another.

60. Two conjugate diameters of an ellipse include an angle γ; show that these diameters are equal to one another when $\sin\gamma$ is the least possible. In this case find the value of γ; a being 8 and b, 5.

61. If CP$=a'$, and CD (the semi-conjugate to CP)$=b'$, have such a position that $\angle$A′CP$=\alpha$, and DCB$=\beta$; show that

$$\frac{a'^2-b'^2}{a^2-b^2}=\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}.$$

62. If C be the centre of an ellipse, and in the normal to any point P, PQ be taken equal to the semi-conjugate at P; show that the locus of Q is a circle of which C is the centre.

It may be shown that CQ=a−b, *a constant quantity, and therefore the radius of a circle about* C.

63. CP and CD are semi-conjugate diameters of an ellipse, and

Ex. 4.

PF is a perpendicular let fall upon CD or CD produced; determine the locus of the point F.

64. The equation to an ellipse being $x^2+3y^2=4$; find the equation to the diameter conjugate to that represented by $2y=3x$.

65. Prove that the distance of P (see figure) from a diameter, drawn parallel to the line joining S and D, is equal to b.

66. If the conjugate semi-axes a', b', of an ellipse, are inclined at angles α, β respectively to the semi-axis major; prove that $a'^2 \sin 2\alpha + b'^2 \sin 2\beta = 0$.

67. In an ellipse, show that $(SP-AC)^2+(SD-AC)^2=SC^2$. (See figure.)

68. If the tangents at the extremities of any diameter DD′ of an ellipse be intersected by the tangent at any other point, in T, T′; then $DT.D'T'=CP^2$, CP being the semi-conjugate to CD.

69. Find the locus of the middle points of chords joining the extremities of conjugate diameters in an ellipse.

70. A triangle is described about an ellipse; prove that the products of the alternate segments of the sides made by the points of contact are equal.

71. If a polygon circumscribe an ellipse, the continued products of the alternate segments are equal to one another.

72. Find the equation to an ellipse when referred to axes which intersect in the centre of the ellipse and make equal angles with the axis major.

73. In an ellipse whose semi-axes are 5 and 4, find the position of CP when an arithmetic mean between CA and CB.

74. If PQ be two points in an ellipse, such that the lines CP, CQ are at right angles to each other, then will $\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

75. If (θ, r), (θ', r') be the coordinates of any two points in an ellipse, the centre being the pole and the major semi-axis the prime radius vector, show that $\frac{1}{r^2}-\frac{1}{r'^2}=\frac{(a^2-b^2)}{a^2b^2}(\sin^2\theta-\sin^2\theta')$.

76. Pp, Qq are chords of an ellipse drawn through one of the foci at right angles to each other; prove that $\frac{1}{Pp}+\frac{1}{Qq}=\frac{a^2+b^2}{2ab^2}$.

77. If P be any point in an ellipse, the vertex of which is A, and nearer focus S; prove that, if $\angle PAS=\theta$, $\angle ASP=\phi$,

$$\tan\theta\tan\tfrac{1}{2}\phi=1+e.$$

78. If a line be drawn through the focus of an ellipse, making an angle θ with the major axis, and tangents be drawn at the extremities of this line, these tangents will be inclined to one another at an angle ϕ, such that $\tan\phi=\frac{2e}{1-e^2}\sin\theta$.

Ex. 4.

79. If tangents drawn to any two points of an ellipse meet each other; show that their lengths are inversely as the sines of the angles which they make with the lines drawn to either focus.

80. Find the locus of the middle points of all focal chords of an ellipse.

81. PSp is any focal chord of an ellipse, A an extremity of the major axis, AP, Ap meet the directrix in Q, q; show that QSq is a right angle.

82. If R be any point in the circumference of a circle described on AA′ the major axis of an ellipse; and AR, A′R be joined, cutting the ellipse in Q, Q′ respectively, prove that $\dfrac{AR}{AQ}+\dfrac{A'R}{A'Q'}=\dfrac{a^2+b^2}{b^2}$.

83. If SQ be drawn always bisecting the angle PSC of an ellipse (see figure) and equal to the mean proportional between SC and SP; find the eccentricity of the locus of Q.

84. If lines, drawn through any point in an ellipse to the extremities of any diameter PCP′, meet the direction of its conjugate diameter DCD′ in M, N; prove that $CM.CN=CD^2$.

85. If r, r' be the distances of the two foci of an ellipse from a point P in the curve; and s, s' from a point D in the curve; P and D being the extremities of conjugate diameters; prove that

$$rr'+ss'=a^2+b^2.$$

86. Find the locus of E, the intersection of SP and CD: (see fig.)

87. Prove that $\cos SPT=e\cos STP$: (see figure).

88. If CD be the semi-diameter of an ellipse drawn parallel to a focal chord PSp; prove that $2CD^2=AC.Pp$.

89. The centre of an ellipse coincides with the vertex of a common parabola, and the axis major of the ellipse is perpendicular to the axis of the parabola. Required the proportion of the axes of the ellipse, so that it may cut the parabola at right angles.

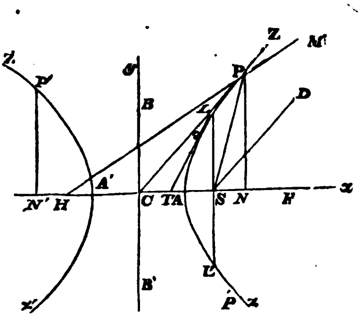
90. If ρ, ρ' be the radii of curvature at the extremities of two conjugate diameters of an ellipse, show that $\rho^{\frac{2}{3}}+\rho'^{\frac{2}{3}}$ is constant. Also if c, c' be the curvatures at two points at which the tangents are at right angles, $c^{\frac{2}{3}}+c'^{\frac{2}{3}}$ is constant.

91. Find expressions for the chords of curvature through the focus and centre of an ellipse.

92. Find the equation to the curve from any point of which if two tangents be drawn to a given ellipse the angle contained between them shall be constant.

V. Hyperbola.

General Properties of the Hyperbola.



If $CA=a$, $CB=b$, $CN=x$, $NP=y$;

(1) $y^2=\frac{b^2}{a^2}(x^2-a^2)$.

(2) LL' *the lat. rect.* $=\frac{2b^2}{a}=2a(e^2-1)$.

(3) $SP=ex-a$, $HP=ex+a$.

(4) $a^2yy'-b^2xx'=-a^2b^2$ *the equation to tangent.*

(5) $CP^2-CD^2=a^2-b^2$, CP, CD *being conjugate semi-diameters.*

(6) $r=\frac{a(e^2-1)}{1-e\cos\theta}$, *if* $SP=r$, $\angle xSP=\theta$.

(7) $4xy=a^2+b^2$, *the equation to the Hyperbola referred to its asymptotes.*

(8) *In an equilateral hyperbola* $a=b$.

Ex. 5. Determine the position and axes of the following curves—

1. $y^2-x^2-y=0$.
2. $y^2-2xy-x^2+2=0$.
3. $4y^2-8xy-4x^2-4y+28x-15=0$.
4. $4xy-3x^2=a^2$.
5. $xy=5x^2-7$.
6. $xy=x^2\tan\alpha+\frac{1}{4}b^2\cot\alpha$.
7. $y^2-10xy+x^2+y+x+1=0$.
8. $y^2-2xy-x^2-y+x+1=0$.
9. $y^2-3xy+x^2+1=0$; the coordinate axes being inclined to each other at an angle $\frac{1}{3}\pi$.

10. The semi-axis CA of an equilateral hyperbola is intersected in T by a tangent to the curve at P: PN is the ordinate at P, and CP is joined; show that CP.PN=PT.CN.

11. An ellipse and a hyperbola have the same foci; show that these curves will intersect one another at right angles.

12. Express the length of a perpendicular, from the centre of a hyperbola upon the tangent at any point, in terms of its inclination to the transverse axis.

13. If A and B be the extremities of the axis major of a hyperbola or an ellipse, T the point where the tangent at P meets AB or AB produced, QTR a line perpendicular to AB and meeting AP and BP in Q and R respectively, then QT=RT.

14. If in a hyperbola, P be a point, whose ordinate $=\left(\frac{BC^3}{SC}\right)^{\frac{1}{2}}$, and CY be a perpendicular from the centre upon the tangent at P; then PY=SC.

Ex. 5.

15. A pair of conjugate hyperbolas being given, to find their centre.

16. In a rectangular hyperbola, show that every diameter is equal to its conjugate diameter.

17. Find the latus rectum and eccentricity of the hyperbola which is conjugate to that whose equation is $y^2=4(x^2+a^2)$.

18. Of two conjugate diameters of a hyperbola, one only meets the curve; and if one be drawn through a given point of the curve, find where the other meets the conjugate hyperbola.

19. If CP, CD be conjugate semi-diameters of a hyperbola, show that, normals being drawn at P, D which intersect in K, KC is perpendicular to PD.

20. If, in a hyperbola, CP, CD be conjugate semi-diameters, and DM, PN be perpendiculars on CA; show that

$$DM : CN = BC : AC = PN : CM.$$

21. Find the area included by the normals to a hyperbola which pass through the foci of the conjugate hyperbola.

22. The tangent at any point of a hyperbola is produced to meet the asymptotes; show that the triangle cut off is of constant magnitude.

23. If a line intersecting a hyperbola in the point P, and its asymptotes in R, r, move parallel to itself, the rectangle RP . Pr is constant.

24. If $3CA=2CS$ in a given hyperbola, find the inclination of the asymptote to the transverse axis.

25. CP is any semi-diameter of a hyperbola: a straight line QRr parallel to PC cuts the curve in Q and the asymptotes in R, r: prove that $QR \cdot Qr = CP^2$.

26. A straight line HPQK cuts the asymptotes Cx, Cy of a hyperbola in the points H, K and the curve in P, Q; prove that $PH=QK$.

27. The radius of the circle, which touches a hyperbola and its asymptotes, is equal to that part of the latus rectum produced, which is intercepted between the curve and the asymptote.

28. Prove that every chord of a hyperbola divides into two equal parts the portion of either asymptote which is included between the tangents at its extremities.

29. Investigate the polar equation to the hyperbola, the focus being the pole, having given $SP-HP=2AC$; and draw the asymptote by means of this equation.

30. If normals be drawn to an ellipse from a given point within it, the points where they cut the curve will all lie in a rectangular hyperbola which passes through the given point and has its asymptotes parallel to the axes of the ellipse.

Ex. 5.

31. If any right-angled triangle be inscribed in an equilateral hyperbola, the perpendicular drawn from the right angle to the hypothenuse is a tangent to the curve.

32. Find the locus of the extremity of a perpendicular from the centre of a hyperbola upon a tangent at any point.

33. Find the locus of the intersection of two tangents to a hyperbola which meet one another at right angles.

34. Find the locus of the centre of the circle inscribed in the triangle SPH; S, H being the foci and P any point in the curve.

35. Find the radius of curvature to a hyperbola at the extremity of its latus rectum, the axes being 20 and 12.

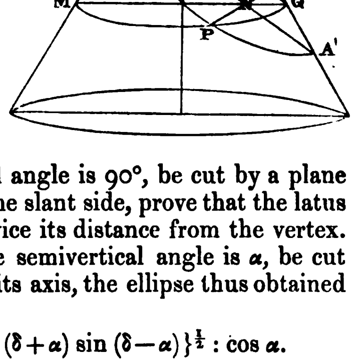
$$\textit{The radius of curvature at any point} = \frac{(\textit{normal})^3}{(\textit{semi-lat. rect.})^2}.$$

Sections of the Cone and General Problems.

If AN$=$x, NP$=$y, AV$=$d, $\angle$VAN$=\theta$, $\angle$AVQ$=2\alpha$;

$$y^2 = \frac{2d\sin\alpha\sin\theta}{\cos\alpha}x - \frac{\sin\theta\sin(2\alpha+\theta)}{\cos^2\alpha}x^2,$$

is the equation to the conic section. Hence $y^2 = 2mx + nx^2$, *may be used as the general equation.*



Ex. 6.

1. If a cone be right-angled, what is the inclination of its axis to the plane of those sections, the eccentricity of which is $\frac{1}{\sqrt{2}}$?

2. If a right cone, whose vertical angle is 90°, be cut by a plane which is parallel to one touching the slant side, prove that the latus rectum of the section is equal to twice its distance from the vertex.

3. If a right cone, of which the semivertical angle is α, be cut by a plane making an angle δ with its axis, the ellipse thus obtained will be such, that the

$$\text{minor axis} : \text{major axis} = \{\sin(\delta+\alpha)\sin(\delta-\alpha)\}^{\frac{1}{2}} : \cos\alpha.$$

4. A section of a right cone is made by a plane parallel to the slant side; when the plane passing through its directrix and the vertex of the cone is perpendicular to the cutting plane, determine the vertical angle of the cone.

5. The section of a right cone by a plane is an ellipse of which a and b are the axes, h and k the distances from the vertex to the

Ex. 6.

points where the plane cuts the sides of the generating triangle; show that $a^2-b^2=(h-k)^2$.

If 2α be the vertical angle of a cone, it may be shown that
$$a^2=(h-k)^2+4hk \sin^2 \alpha, \text{ and } b^2=4hk \sin^2 \alpha.$$

6. When the section of a cone is an ellipse or a hyperbola, show that the semi-minor axis is a mean proportional between the perpendiculars dropped from the extremities of the major axis upon the axis of the cone.

7. Find the latus rectum of the section of a right cone made by a plane parallel to the slant side; and when the plane passing through the directrix of the section, and the vertex of the cone is perpendicular to the cutting plane, determine the vertical angle of the cone.

The vertical angle is equal to $\cos^{-1}(\frac{1}{3})$.

8. At what angle must a plane be inclined to the side of a cone in order that the section may be a rectangular hyperbola? Show that the least vertical angle of the cone must be greater than the angle between the asymptotes.

9. If a right cone be cut by a plane in a hyperbola, the angle ϕ between the asymptotes of the section is determined by the equation $\cos\frac{\phi}{2}=\frac{\cos\alpha}{\sin\theta}$, where α is the semivertical angle of the cone, and θ the inclination of the cutting plane to the base of the cone.

10. If the length of the axis of an oblique cone be equal to the radius of its base, every section perpendicular to the axis will be a circle.

11. The tangents to the interior of two concentric and similar curves of the second order, whose axes are coincident, cut off from the exterior curve equal areas.

12. Prove that in any conic section the diameter of curvature varies as the cube of the normal.

13. In any conic section the projection of the normal on the focal distance is constant.

14. In any conic section, if S be the focus and G the foot of the normal at P, then SG varies as SP.

15. The sum of the squares of normals at the extremities of conjugate diameters in any conic section is constant.

16. AP is the arc of a conic section of which the vertex is A; PG the normal, and PK a perpendicular to the chord AP, meet the axis in G and K respectively. Show that GK is equal to half the latus rectum.

17. If S be the focus and A the vertex of any conic section, and

Ex. 6.

if LT the tangent at the extremity L of the latus rectum meet the axis in T, show that $\frac{AS}{AT}$ equals the eccentricity.

18. In any conic section, if two chords move parallel to themselves and intersect each other, the ratio of the rectangles of their segments is invariable.

19. Find the length of the chord of a conic section, the equations to the chord and section being $\frac{x}{h}+\frac{y}{k}=1$, and $y^2=2mx+nx^2$.

20. From the extremities of any chord PQ of a conic section, draw the tangents PT, QT to meet in T; then if α be the angle which the chord PQ makes with its corresponding diameter,

$$\cot TPQ-\cot TQP=2\cot\alpha.$$

21. If PSp be any straight line drawn through the focus S of a conic section, meeting the curve in the points P and p, and SL be the semi-latus rectum, then will SP, SL and Sp be in harmonical progression.

22. A straight line drawn from the intersection of two tangents to a curve of the second order is harmonically divided by the curve and the chord joining the points of contact.

23. A conic section is cut in four points by a circle, and two lines each passing through two of the points of intersection are made the axes of coordinates, their point of meeting being the origin; show that the equation to this conic section is of the form

$$x^2+bxy+y^2+dx+ey+f=0.$$

Under what conditions is the converse true?

24. The distance of a point P from the circumference of a circle : its distance from a fixed diameter AB : : n : 1. Prove that the locus of P is a conic section.

25. If two lines revolving in the same plane round the points S and H, intersect one another in the point P in such a manner that, (1) SP^2+HP^2 equals a constant quantity; (2) SP is to HP in the given ratio of n : 1; prove that in each case the locus of P is a circle.

26. Find the locus of the intersection of two normals to a curve of the second order, which cut one another at right angles.

27. Find the locus of the intersection of two tangents to a curve of the second order, which cut one another at any given angle.

28. Find the equation of the curve traced out by the extremities of the perpendiculars on the tangents of a circle, drawn from a point in its circumference.

Ex. 7.

1. Determine what the equation $5(x^2+y^2)-6xy=4c^2$ becomes, when the axes are turned through an angle of 45°.

Ex. 7.

2. If the coordinate axes of the curve whose equation is $(x^2-y^2)^2=ax(x^2+3y^2)$ be made to revolve about the origin through an angle of $45°$; required the equation to the curve referred to the new axes.

3. Determine the axes to which the rectangular equation $a^2y^2+b^2x^2=a^2b^2$ must be transferred, so that the transformed equation may be $x'^2+y'^2=a'^2$.

4. The equation $y^4+2a^2xy-x^4=0$ expressed by polar coordinates is $r^2=a^2\tan 2\theta$.

Loci.

Ex. 8.

1. Find the equation to the locus of a point the difference of whose distances from two fixed points is invariable; and trace the curve.

2. The base of a triangle is constant, and the sum of the angles at the base is also constant; find the locus of the vertex.

3. Having given one side of a triangle, and the difference between the tangents of the adjacent angles; find the locus of the vertex.

4. The base of a triangle is given, and one angle at the base is double of the angle at the other extremity of the base; find the locus of the vertex.

5. Determine the locus of a point so situated within a plane triangle, that the sum of the squares of the straight lines drawn from it to the angular points is constant; if the curve has a centre, find its position.

6. Having given the base and area of a triangle, find the locus of the centre of the circumscribed circle.

7. Straight lines are drawn from a fixed point to the several points of a line given in position, and on each as a base is described a triangle whose vertical angle is one-half of each of the angles at the base; find the locus of these vertices.

8. The base of a triangle and the sum of the other two sides are given; find the locus of the centre of the inscribed circle.

9. Having given the base and altitude of a triangle, to find the locus of the centre of the inscribed circle.

10. Find the locus of the vertex of a triangle upon a given base, and having its vertical angle bisected by a line parallel to a given line.

11. If the base and the difference of the angles at the base of a triangle be given, the locus of the vertex is an equilateral hyperbola.

12. The corner of a page is turned down, so that the triangle is of a constant area a^2; the locus of the angular point is a lemniscata whose equation is $r^2=a^2\sin 2\theta$.

13. If two lines SP, HP revolve about the points S, H so that

Ex. 8.

$SP \times HP = CS^2$ (C being the middle point of SH); it is required to find the locus of P.

14. If from two fixed points in the circumference of a circle, straight lines be drawn intercepting a given arc and meeting without the circle; to find the locus of their intersection.

15. A straight line revolving in its own plane about a given point intersects a curve line in two points; find the curve when the rectangle of the lines intercepted between the given point and the points of intersection is constant.

16. A straight line of given length $2c$ is made to move so that its ends are always in contact with two other straight lines which include a given angle 2α; show that the locus of its middle point is an ellipse whose semi-axes are $c \tan \alpha$ and $c \cot \alpha$; and the direction of one of its axes bisects the angle included between the two given lines.

17. If an ellipse be moved between two straight lines at right angles to one another, to show that the centre will describe a circle; and to find the locus of any given point in the axis.

18. If a parabola be moved between two straight lines at right angles to one another, the equation to the locus of its vertex will be $x^{\frac{4}{3}}y^{\frac{2}{3}} + x^{\frac{2}{3}}y^{\frac{4}{3}} = a^2$.

19. The chord of contact of two tangents to a parabola subtends an angle β at the vertex; show that the locus of their point of intersection is a hyperbola whose asymptotes are inclined to the axis of the parabola at an angle ϕ, such that $2 \tan \phi = \tan \beta$.

20. From two given points A and B two straight lines given in position are drawn; MRQ is a common ordinate to these lines, and MP is taken in MRQ a mean proportional to MQ and MR; required the locus of P.

21. Let AQA′ be an ellipse, AA′ the axis major, QQ′ any double ordinate; join AQ and A′Q′ and produce these lines to intersect one another in P; the locus of P is required.

22. To find the locus of the centres of all the circles drawn tangential to a given line, and whose circumferences pass through a given point.

23. Let AQB be a semicircle of which AB is the diameter, BR an indefinite straight line perpendicular to AB, AQR a straight line meeting the circle in Q and BR in R; take AP=QR; required the locus of P.

The locus of P *is called the Cissoid of Diocles.*

24. A point Q is taken in the ordinate MP of the parabola, always equidistant from P and from the vertex of the parabola; required the locus of Q.

Ex. 8.

25. Let AQB be a semicircle in which AB is the diameter and NQ is any ordinate produced to P, so that NP : NQ = AB : AN; to find the locus of P.

The locus of P *is called the Witch of Agnesi.*

26. Let XX′ be an indefinite straight line, A a given point without it, from which draw the straight line ACB perpendicular to XX′ which it cuts in the point C, and also any number of straight lines AEP, AE′P′, &c.; take EP, always equal to CB; then the locus of P is required.

This locus is called the Conchoid of Nicomedes.

27. If P be a point in a cycloid and O the corresponding position of the centre of the generating circle, show that PO touches another cycloid of half the dimensions.

Ex. 9. Trace the curves whose equations are,—

1. $r = 2a \sin\theta \tan\theta$.
2. $r^2 = a^2 \cos 2\theta$.
3. $y^3 = a^2x - x^3$.
4. $y + x = (3x)^{\frac{1}{2}}$.
5. $\dfrac{y^2}{x^2} = \dfrac{a^2 + x^2}{a^2 - x^2}$.
6. $y = \dfrac{x - 3}{(x - 1)(x - 2)}$.
7. $y^2 = 3x \pm x(9 - x^2)^{\frac{1}{2}}$.
8. $(x^2 + y^2)\{x - (x^2 - y^2)^{\frac{1}{2}}\} = ay^2$.
9. $xy = \pm(a + y)(b^2 - y^2)^{\frac{1}{2}}$, where b is $> a$.

STATICS.

Forces acting in the same Plane and whose Directions (produced if necessary) pass through one Point.

The two conditions of equilibrium are,—

1. *The sum of the resolved forces in* any *direction* $=0$.
2. *The sum of the resolved forces at right angles to the first set* $=0$.

Ex. 1.

1. Two forces represented by 2, 3 are inclined to each other at an angle of 45°: find their resultant.

2. At what angle must two forces 3, 4 act, so that their resultant may be 5?

3. If two forces be inclined to each other at an angle of 135°; find the ratio between them, when the resultant equals the less.

4. If two forces acting on a point be in the ratio of 2 to 3; find the angle between them, when their resultant is a mean proportional between them.

5. Two forces, which are to each other as 2 to $\sqrt{3}$, when compounded, produced a force equivalent to half the greater. Find the angle at which they are inclined to one another.

6. The resultant of two forces is 10 lb.; one of them is 8 lb., and the other is inclined to the resultant at an angle of 36°: find the other force, and the angle between the two.

7. Three forces acting on a point, and in the same plane, are in equilibrium when their directions are inclined to each other at angles of 60°, 135°, and 165° respectively. Find the ratio of the forces.

8. Three forces acting on a point, keep it at rest; and they are in the ratios of $\sqrt{3}+1 : \sqrt{6} : 2$. Find the angles at which they are respectively inclined to each other.

9. Three equal forces, each equivalent to 6 lb., act on a point; the first two are inclined to each other at an angle of 75°, and the third is inclined at an angle of 15° to the first. Find the magnitude and direction of the resultant.

10. Four forces, represented by 1, 2, 3 and 4, act on a point. The directions of the first and third are at right angles to each other; and so are the directions of the second and fourth; and the second is inclined at an angle of 60° to the first. Find the magnitude and direction of the resultant.

11. Three forces act perpendicularly to the sides of a triangle at the middle points, and each is proportional to the side on which it acts. Show that they will keep each other at rest.

Ex. 1.

12. If forces, proportional to the sides of any polygon, act perpendicularly to these sides respectively at their middle points, they will keep each other at rest.

13. If three forces A, B, C act upon a given point, and keep it at rest; given the magnitude and direction of A, the magnitude of C and the direction of B: determine the magnitude of B and the direction of C.

14. A point in the vertex of a right-angled triangle is acted on by a number of forces, represented in magnitude and direction by lines drawn to equidistant points in the base: find the resultant.

15. Resolve the force P into two others p, p', which shall act at a given angle α, and whose difference d is given.

16. Resolve a force represented by 20 into two others, whose sum is 22, and which contain an angle of 60°.

17. Two chords AB, AC of a circle represent two forces; if AB be given, find the position of the other when the resultant is a maximum.

18. If R be the resultant of the forces P and Q acting in the same plane on a point A, and r, p, q the respective distances of A from the perpendiculars drawn from any point in the plane on the direction of R, P, and Q respectively, then shall $Rr = Pp + Qq$.

19. The angles A, B, C of a triangle are 30°, 60°, 90° respectively. The point C is acted on by forces in directions CA, CB inversely proportional to CA, CB; find the magnitude and direction of their resultant.

20. Two forces 3 and 4, act on the ends of a rigid rod 10 feet long, and the angles included between their directions and the rod are 30° and 60° respectively. Find the magnitude and position of a force which shall keep the rod at rest.

21. If two forces P, Q be represented in magnitude and position by the focal distances of the extremities of the major axis of an ellipse, which are as 1 : 3; and another force R, by the focal distance of an extremity of the minor axis: show that the resultant of P and R is equal to $(P \cdot Q)^{\frac{1}{2}}$.

When a body is supported by means of a string, the tension (T) *of the string must be considered as one of the forces that preserve equilibrium. In the same string the tension is the same at every point and in both directions. When one string only supports a weight, the tension of the string equals the weight.*

Ex. 2.

1. A weight of 50 lb., suspended freely from a fixed point A, is drawn aside from the vertical through an angle of 45° by a force acting horizontally; determine this force and the pressure on A.

2. Three weights of 4, 5, 6 lb. respectively, are suspended over the circumference of a circular hoop, by three strings knotted together

Ex. 2.

at its centre: determine the relative positions of the strings, when the hoop supported at its centre remains horizontal.

3. Two equal weights P are connected by a string passing over two fixed pulleys A and B, situated in a horizontal line, and support a weight W $(=P\sqrt{3})$ hanging from a ring C, which slides freely on the string AB. Find the position of equilibrium.

4. A weight W is suspended from one extremity of a string, which passes through a ring C fastened at its other extremity; the string passing over two fixed pulleys, A and B, in the same horizontal line. Find the position of equilibrium.

5. A weight W is suspended from one extremity of a string which passing through a smooth small ring at B is fastened at its other extremity to a fixed point A in the same horizontal line with B; at C, a fixed point of the string, another weight P is suspended; find the ratio of P to W that the vertical line through C may bisect AB.

6. A and B are two given points in a horizontal line; to A a string AC is fastened $=\frac{1}{2}$AB; to B another string is fastened, which passing through a ring at C, supports a weight at its other extremity. Find the position of equilibrium.

7. A weight W is supported by two strings, which passing over two fixed pulleys A, B, not in the same horizontal line, have two weights P, Q at their other extremities; find the position of W when at rest.

8. Two equal weights are suspended by a string passing freely over three tacks, which form an isosceles triangle, whose base is horizontal, and vertical angle $=120°$. Find the pressure on each of the tacks.

9. If a string ACDB be 21 inches long; C and D two points in it, such that $AC=6$, $CD=7$; and if the extremities A and B be fastened to two fixed points in the same horizontal line at a distance of 14 inches from each other; what must be the ratio of two weights, which hung at C and D, will keep CD horizontal?

10. A string ACDB, of which the extremities A, B are fixed, supports the weights P, W suspended at knots C, D; if AC, BD be produced to meet the directions of the weights in d, c respectively; show that $P : W = Dd : Cc$, when there is equilibrium.

11. A string ABCDE, of which the extremities A, E are fixed, is kept in a given position by the weights P, Q, R suspended at knots B, C, D respectively; compare the weights P, Q, R.

12. A weight W is sustained upon a smooth inclined plane by three forces, each equal to $\frac{1}{3}$W, which act, one vertically upwards, another horizontally, and the third parallel to the plane; find the inclination of the plane to the horizon.

13. Two forces 5, 3 acting respectively parallel to the base and

Ex. 2.

length of an inclined plane will, each of them singly, sustain upon it a weight W; determine the magnitude of W.

14. Two weights P, Q are connected together by a string passing over a smooth fixed pulley E; P rests on a smooth inclined plane AB, and Q hangs freely; determine the position of equilibrium, and the pressure on the plane.

15. Two weights, P and Q, support each other on two planes, inclined to the horizon at angles α and β respectively, by means of a string passing over the common vertex of the planes. Find the ratio of P to Q, and the tension of the string.

16. O is the centre, OA a horizontal radius, of a quadrant; over a pulley at A a string is passed supporting two weights P and 2P at its extremities, the former of which hangs vertically, while the latter rests on the arc of the quadrant. Find the position of equilibrium.

Take the moments about O *of the forces acting on* (2P).

17. Over a vertical semicircle ABD, whose centre is C, a string is laid, which is equal in length to the arc of a quadrant of the circle; and which has two weights, P and Q, at its extremities. Find the angle PCA, when the position is one of equilibrium.

Take the moments of P, Q *about* C.

18. A string passes over a pulley at the focus of a parabola, whose axis is vertical; from one extremity of the string a weight P hangs vertically; and at the other extremity a weight Q rests on the convex side of the parabola. Show that $P=Q$; and that equilibrium exists in all positions.

19. A sphere, of weight W, rests on two planes inclined at angles α and β to the horizon. Find the normal pressures on the planes.

The weight of the sphere is one of the forces, and is supposed to act at its centre, vertically downwards.

Forces acting in one Plane, but not through the same Point.

The three conditions of equilibrium are,—

1. *The sum of the resolved forces in* any *direction* $=0$.
2. *Their sum in a direction at right angles with the first direction* $=0$.
3. *The sum of the moments about* any *point* $=0$.

In determining these directions, and the position of this point, the solution is in most cases rendered more simple by resolving the forces in a direction perpendicular to the direction of some force that is not known nor required, in order that this force may not appear in the equation, and by taking the moments of the forces about a point through

which this unknown force passes, in order that it may not appear in another equation. And thus two equations will be all that are required, instead of three, which must otherwise be employed.

The tensions of strings, reactions of planes, weights of bodies, &c., are considered as forces.

Ex. 3.

1. If two parallel forces act in the same direction along the opposite sides AB, DC of a parallelogram, and another force act along the diagonal BD; and if these three forces be respectively proportional to AB, DC, and BD; find the magnitude and position of a fourth force which will keep the parallelogram at rest.

2. Two forces F, F′ acting in the diagonals of a parallelogram whose weight is W, keep it at rest in such a position that one of its edges is horizontal: if α, α' be the angles between the diagonals and the horizontal side; $F \sec \alpha = F' \sec \alpha' = W \operatorname{cosec} (\alpha + \alpha')$.

3. AB is a rod capable of turning freely about its extremity A, which is fixed; CD is another rod equal to 2AB and attached at its middle point to the extremity B of the former, so as to turn freely about this point; a given force P acts at C in the direction CA; find the force Q which must be applied at D to produce equilibrium.

4. If a set of forces, acting at the angular points of a plane polygon, be represented by the sides taken in order, show that their tendency to turn a body about an axis perpendicular to the plane of the polygon is the same, through whatever point of the plane the axis passes.

5. ABC is an isosceles triangle, C being a right angle; and three equal forces act in the lines AB, BC, CA. Show that their resultant is to one of the forces as $\sqrt{2} - 1 : 1$; and that if CD be drawn perpendicular to AB, and DC produced to E, so that $DE = CE\sqrt{2}$; then the resultant acts through E in a direction parallel to BA.

6. A string, having its extremities fixed to the ends of a uniform rod, of weight W, passes over 4 tacks, so as to form a regular hexagon; the rod (which is horizontal) being one of the sides; find the tension of the string and the vertical pressure on each tack.

7. A uniform beam PQ hangs by two strings AP, BQ, from any two fixed points A and B; when there is equilibrium, compare the tensions of the strings with each other, and with the weight of the beam.

8. A uniform beam rests on two planes inclined at angles α and β to the horizon; find the inclination θ of the beam to the horizon; and the pressures on the two planes.

9. Two spheres rest upon two smooth inclined planes, and press against each other: determine the inclination to the horizon of the line joining their centres.

10. Two spheres are at rest (pressing against each other) in a

Ex. 3.

given hemispherical bowl whose axis remains vertical; determine θ, the inclination to the horizon of the line joining their centres.

11. A uniform rod AB is placed with one end A, inside a hemispherical bowl (whose axis remains vertical), and at a point P rests on the edge of the bowl: if $AB = 3 \times$ radius, find AP.

Resolve parallel to the rod, and take moments about P.

12. A uniform beam AB rests with one end A on a smooth vertical wall; the other end B is supported by a string fastened to a point C in the wall. If the length of the beam be 3 feet, and the length of the string 5 feet; find CA, and the tension of the string.

Resolve vertically, and take moments about A.

13. A uniform beam AB, of weight W, rests with one end A on a horizontal plane AC, and the other end on a plane CB, whose inclination to the horizon is 60°. If a string CA, equal to CB, prevent the beam from sliding, what is the tension of this string?

Resolve horizontally, and take moments about A.

14. A uniform beam AB, whose weight is W, and length 6 feet, rests on a vertical prop CD equal to 3 feet; the other end A is on the horizontal plane AD, and is prevented from sliding by a string DA equal to 4 feet. Find the tension of this string.

Resolve horizontally, and take moments about A.

15. One end of a beam, whose weight is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests on a smooth inclined plane (α); the string, passing over a pulley at the top of the inclined plane, hangs vertically supporting a weight P. Find the relation between P, W, and α, when the beam will rest at all inclinations to the horizon, less than α.

16. A uniform beam rests with one end upon a given inclined plane (α), the other end being suspended by a string from a fixed point above the plane; determine the position of equilibrium, the tension of the string, and the pressure of the plane.

17. A uniform beam AB has two strings fastened to its ends, one of which, AC equal to length of beam, is fastened to a ring C; and *through* C the other string BCP passes, supporting a weight P equal to half the weight of AB. If A be the lower end of the beam, find its inclination to the horizon.

Resolve perpendicularly to AC, *and take moments about* A.

18. A uniform beam AB rests with one end A on a prop; to the other end B is fixed a string which passes over a pulley D, at the distance $AD = AB$, and sustains a weight P at its other extremity; determine the position of equilibrium.

Ex. 3.

19. Two unequal weights P, Q connected by a rigid rod without weight, are suspended by a string fastened at the extremities of the rod, and passing over a fixed point; determine the positions of equilibrium.

20. A rigid rod AB rests upon a fixed point D, while its lower extremity A presses against a vertical wall EF; find the position of equilibrium, and the pressures at A and D.

21. A uniform beam AB is placed in a vertical plane, with one end A on a horizontal plane CA, and the other end B against a vertical plane CB; the beam is now kept at rest by a string EC, E being a given point in AB: find the tension of the string.

22. A uniform beam, 6 feet in length, rests with one end against a smooth vertical wall, the other end resting on a smooth horizontal plane, and is prevented from sliding by a horizontal force, applied at that end, equal to the weight of the beam, and by a weight equal to $\frac{2}{3}$ the weight of the beam, suspended from a certain point on the beam. Find the distance of this point from the lower end of the beam, if it be inclined at an angle of 45° to the horizon.

Resolve horizontally, and take moments about the lower end of beam.

23. To find the position of equilibrium of a uniform beam, one end of which rests against a smooth vertical plane, and the other on the interior surface of a given hemisphere.

Let θ be the inclination of the beam to the horizon, and ϕ of the radius at the point where the beam presses against the hemisphere: resolve vertically and horizontally, and take the moments about the lower end of the beam.

24. A uniform beam AB rests with one end on a horizontal plane AC, and the other on the convex surface of a hemisphere, whose centre is C: determine the horizontal force which must be applied to keep the beam in a given position, and the pressures on the sphere and plane.

25. A given weight W is held at rest on the convex arc of a circular quadrant lying in a vertical plane, by means of a given weight Q acting over a pulley B; B, and C the centre of the quadrant, being in the same vertical line: required the position of rest.

26. Two beams, whose weights are proportional to their lengths 9 feet and 7 feet, rest against each other on a smooth horizontal plane; the upper ends resting against two smooth vertical and parallel walls. If 10 feet be the distance between the walls, determine θ, θ' the inclinations of the beams to the horizon.

27. A uniform lever, whose arms, of lengths $2a$ and $2b$, are at right angles to each other, touches the circumference of a circle,

Ex. 3.

whose plane is vertical, and radius c. Find the inclination of the arm $2a$ to the horizon.

Resolve in direction of one arm, and take moments about the point of contact of the same, so that one of the reactions may disappear.

28. Two equal uniform beams AB, AC moveable about a hinge at A, are placed upon the convex circumference of a circle in a vertical plane: find their inclination to each other when at rest.

29. Three uniform beams, AB, BC, CD, of the same thickness, and of lengths l, $2l$, and l respectively, are connected by hinges at B and C, and rest on a perfectly smooth sphere, the radius of which $=2l$; so that the middle point of BC, and the extremities A and D are in contact with the sphere. Find the pressure at the middle point of BC.

Resolve the 6 forces vertically; then, considering AB *kept at rest by two forces about a fixed point* B, *take the moments of these forces about this point.*

30. A roof ACB consists of beams which form an isosceles triangle, of which the base AB is horizontal. Given W the weight of each beam, and α the angle at which it is inclined to the horizon; find the force necessary to counterbalance the horizontal thrust at A.

Take the moments about C.

31. A uniform beam of length $2a$, moveable in a vertical plane about a hinge at A, leans upon a prop of length b situated in the same plane: determine the strain upon the prop, α, β being the inclinations to the horizon of the beam and prop respectively.

32. A uniform isosceles triangle, of which a is the length of each of the equal sides and h the altitude, is placed in a smooth hemispherical bowl of radius r, its three angles touching the bowl; find the position in which it will rest.

The centre of gravity of the triangle is in the line h *at a distance from the base equal to* $\frac{1}{3}$h.

33. A uniform rod of length $2a$ rests against a peg at the focus of a parabola, its lower extremity being supported on the curve; if $4m$ be the latus rectum of the parabola whose axis is vertical; determine the inclination of the rod to the horizon.

Resolve in direction of a tangent at the point where the rod rests on the curve, and take moments about that point.

34. A smooth sphere, of radius 9 inches, and weight 4 lb., is kept at rest on a smooth plane, inclined at an angle of 30° to the horizon, by means of a uniform beam, of length 7 feet, moveable about a hinge on the plane, and resting on the sphere. Find what

Ex. 3.

must be the weight of the beam, that it may be inclined at an angle of 15° to the plane.

Resolve parallel to the plane; the beam is kept at rest round a fixed point by two forces; its own weight, and the reaction of the sphere.

35. A cylinder with its axis horizontal, is supported on an inclined plane by a beam, which rests upon it and has its lower extremity fastened to the plane by a hinge; find the conditions of equilibrium.

36. A given weight P is suspended from the rim of a uniform hemispherical bowl of weight W placed on a horizontal plane: to find the position in which the bowl will rest.

If the body have a fixed *point, the only necessary condition of equilibrium is that—*

The sum of the moments of all the forces about the point $=0$.

37. At what point of a tree must a rope of given length a be fixed, so that a man pulling at the other end may exert the greatest force in upsetting it?

Find the greatest moment about the foot of a tree.

38. A uniform bent lever ACB hangs freely by one extremity A. If C be a right angle, $AC=2a$, $BC=2b$; find the inclinations of AC to the horizon.

39. AC and BC are two uniform rods of equal lengths joined at C, and perpendicular to each other in a vertical plane; but the weight of BC : that of $AC=\sqrt{3}:1$. At what angle will BC be inclined to the horizon, when the angular point C rests on a horizontal plane, and the whole is kept in equilibrium?

40. One end of a beam is connected with a horizontal plane by means of a hinge, about which the beam can revolve in a vertical plane; the other end is attached to a weight equal to 3 times weight of beam by means of a string passing over a pulley in a vertical wall. If the length of the beam = the distance of the hinge from the wall = the height of the pulley above the plane; find the inclination of the beam θ, and of the string ϕ to the horizon.

41. A uniform beam AB is moveable in a vertical plane about a hinge at A; to the other end B a string is attached, which passing over a fixed pulley at C ($AC=AB$) supports a weight = half the weight of the beam. Find the inclination of the beam to the horizon, when AC is vertical.

42. A heavy rod $AB=a$ is moveable in a vertical plane about a hinge at A, and supports with its other extremity B another heavy rod $CD=b$, moveable in the same plane about a hinge at C. If

Ex. 3.

AC $= c$ be horizontal, what must be the ratio between the weights of the two beams, that CB may equal AB?

Consider separately the force acting on the two beams.

43. P and Q are weights fixed to the extremities of a circular arc whose chord $= 2a$, and height $= b$, and which is placed with its plane vertical on a plane inclined at an angle α to the horizon. Find the ratio of P to Q, in order that the arc (prevented from sliding) may rest with its chord parallel to the plane.

Suppose the point of contact of the arc and plane to be fixed, and take moments of P *and* Q *about that point.*

44. A straight uniform rod AC, of 12 lb. weight, and moveable in a vertical plane about a hinge at C, has two equal weights of 2 lb. each, suspended one from the extremity A, and the other from the middle point B; and is kept at rest by a string attached at A, passing over a fixed pulley D, and supporting a weight of 6 lb. If CD = CA, be horizontal; find the inclination of the rod to the horizon.

Forces which do not act in the same Plane.

Take any 3 *lines at right angles to each other, which call the axes of* x, y *and* z*; then*

(1) *The sums of the resolved forces in the directions of the axes of* x, y *and* z *are* separately $= 0$.

(2) *The sums of the moments of the forces about the axes of* x, y *and* z *are* separately $= 0$.

Ex. 4.

1. A right-angled triangle, whose sides are 3, 4 and 5, without weight, rests horizontally on three props placed at its angular points. Find the distances of a point in its plane from the sides containing the right angle, on which if a weight be placed, the pressure at each prop may be proportional to the opposite side.

Take the sides containing the right angle for the axes of x *and* y. *Resolve parallel to axis of* z, *and take moments about axes of* x, y.

2. Any triangle is supported at its angular points, and a weight is laid on it at its centre of gravity. Show that the pressures at the three props are equal.

3. A heavy triangle of uniform thickness and density, is supported in any position by three vertical strings fastened to the angular points. Show that each string supports an equal portion of the weight.

Take the moments about one of the sides.

THE CENTRE OF GRAVITY.

Ex. 5.

1. If G be the centre of gravity of the triangle ABC, then $3(GA^2+GB^2+GC^2)=AB^2+AC^2+BC^2$.

2. If three forces represented in magnitude and direction by three lines GA, GB, GC keep the point G at rest; show that this point coincides with the centre of gravity of the triangle formed by joining the extremities of the lines.

3. Find the centre of gravity of 3 particles of equal weight, placed at the three angles of an isosceles right-angled triangle, whose hypothenuse is equal to 8.

4. The sides of an isosceles triangle are 20, 20, 12; required the distance, from the vertex, of the centre of gravity of the sides.

5. Three particles A, B, C, whose weights are proportional to 3, 2, 1 respectively, are placed so that AB = 5 feet, BC = 4, CA = 2; find the distance of their common centre of gravity from C.

6. If p, q, r be the distances of three particles, whose weights are P, Q, R, from their common centre of gravity, and α, β, γ be the angles contained by the lines p, q; q, r; r, p; respectively: determine the ratios P : Q : R.

7. If the sides of a triangle ABC be bisected in the points P, Q, R, then the centre of the circle inscribed in the triangle PQR is the centre of gravity of the perimeter of ABC.

8. What is the form of a triangle, if its centre of gravity coincide with the centre of a circle circumscribing it?

9. What is the form of a triangle, if its centre of gravity coincide with the centre of a circle inscribed in it?

10. If two isosceles triangles, whose altitudes are h, h', stand upon the same base; find the distance from the base, of the centre of gravity of the area included between the sides of the triangles, when both are situated (1) on the same side of the base, and (2) on opposite sides.

11. One corner of a triangle, equal to an nth part of its area, is cut off by a line parallel to its base; find the centre of gravity of the remaining area.

12. If a, b be the two parallel sides of a trapezoid, its centre of gravity will divide a perpendicular to those sides into two parts that are to each other as $2a+b : a+2b$.

13. At the angles of a square, whose side is 20 inches, are placed particles whose weights are as 1, 3, 5, 7; find the distance of the centre of gravity from the particle of least weight.

14. From a given square, it is required to cut out a triangle, having one side of the square for its base, so that the centre of gravity of the remaining portion may be at the vertex of the triangle.

15. Find the centre of gravity of the figure formed by cutting off

Ex. 5.

one corner from a square, whose side is 14 inches, by a line which bisects the containing sides.

16. Having given a rectangle of uniform thickness, it is required to draw from one of its angles, a line cutting off a triangle, so that the remaining trapezoid, when suspended by the obtuse angle, may hang with its parallel sides horizontal.

17. Find the perpendicular distance of the centre of gravity of any quadrilateral figure from either of the diagonals.

18. Find the distance of the centre of gravity of 3 squares described on the sides of a right-angled triangle, from the right angle; the sides about which are 6 and 8 inches.

19. Two arms of a bent lever are a and b, and the angle between them is α; find the distance of the centre of gravity of the whole from the angular point, the arms being uniform.

20. A table of uniform thickness, and in the form of a regular hexagon, is supported at a point in the under surface, and weights of 7, 11, 15, 19, 23 and 27 lb. are suspended in successive order from the corners of the table; determine the point of support.

21. Find the centre of gravity of five equal heavy particles placed at five of the angular points of a regular hexagon.

22. If PA, PB, PC represent three forces in one plane acting upon a point P; show that the resultant passes through the centre of gravity of the triangle ABC.

23. If on the given chord of a circle as base, triangles are inscribed in the circle: it is required to find the locus of their centres of gravity.

24. Find the centre of gravity of a system of 4 equal bodies, whose centres of gravity are at the corners of a triangular pyramid.

25. A cone, of which the weight is W, rests upon two props placed under the extremities of its axis, which is horizontal; determine the pressure upon each prop.

26. In any triangular pyramid, prove that 4 times the sum of the squares of the distances of the angular points from the centre of gravity of the pyramid, is equal to the sum of the squares of its edges.

27. Find the distance of the centre of gravity of the frustum of a cone from the base; a and b being the radii of the two ends, and c the altitude of the frustum.

28. If two cones have the same base, and their vertices towards the same parts; find the distance of the centre of gravity of the solid contained between their two surfaces from their common base.

29. If two spheres of radii a and b, touch each other internally; find the distance of the centre of gravity of the solid contained between the two surfaces, from the point of contact.

30. Find the distance of the centre of gravity of a hemispherical

Ex. 5.

bowl from the base, a being the internal radius, and c the thickness. Thence find the centre of gravity of a hemispherical surface.

31. Find the centre of gravity of the solid intercepted between the surfaces of a hemisphere and paraboloid on the same base, the latus rectum of the paraboloid coinciding with the diameter of the hemisphere.

N.B. In a paraboloid, $\bar{x}=\frac{2}{3}(axis)$; *in a hemisphere,* $\bar{x}=\frac{5}{8}$ *(radius).*

32. From a cube is cut off a pyramid which has for its base the triangle formed by the diagonals of three adjacent square faces; find the distance of the centre of gravity of the remaining solid from the angle of the cube opposite to the angle cut off.

33. A cube is truncated on one angle by a plane which bisects three adjacent edges; find the position of its centre of gravity.

The integral calculus is required in **Ex. 6.**

Let dm denote an element of the mass of a body at any point x, y, z, referred to any three coordinate axes, and let $\bar{x}, \bar{y}, \bar{z}$ be the coordinates of the centre of gravity of the body: then

$$\bar{x}=\frac{\int x dm}{\int dm}, \qquad \bar{y}=\frac{\int y dm}{\int dm}, \qquad \bar{z}=\frac{\int z dm}{\int dm};$$

the limits of integrations being determined by the form of the body.

1. *For a Symmetrical Area; the formula used is*

$$\bar{x}=\frac{\int xy dx}{\int y dx}: \quad \text{or } \bar{x}=\frac{\iint r^2 \cos\theta \, d\theta \, dr}{\iint r d\theta dr}.$$

Ex. 6. Find the centre of gravity of the area—

1. Of a semicircle.
2. Of a semi-ellipse, the bisecting line being the minor axis.
3. Of a parabola whose axis is a.
4. Of any portion of the parabola whose equation is $ay^2=x^3$, contained by the curve and a double ordinate.
5. Of a cycloid, whose equation is $y=a \text{ vers}^{-1}\frac{x}{a}+(2ax-x^2)^{\frac{1}{2}}$.
6. Of the Cissoid of Diocles, expressed by $y^2(a-x)=x^3$.
7. Of a Catenary; $x=\frac{c}{2}(\epsilon^{\frac{y}{c}}+\epsilon^{-\frac{y}{c}})$, from $x=c$ to $x=\frac{5}{4}c$.
8. Of a given sector of a circle.
9. Of a loop of the Lemniscata, $r^2=a^2\cos 2\theta$.

Ex. 6.

2. *For an Area not Symmetrical; the formulæ are*

$$\bar{x}=\frac{\iint x dx dy}{\iint dx dy},\quad \bar{y}=\frac{\iint y dx dy}{\iint dx y};\quad \text{or } \bar{x}=\frac{\iint r^2\cos\theta d\theta dr}{\iint r d\theta dr},\quad \bar{y}=\frac{\iint r^2\sin\theta d\theta dr}{\iint r d\theta dr}.$$

Find the centre of gravity of the area—

10. Of a quadrant of a circle; $x^2+y^2=a^2$.

11. Of the parabolic figure, bounded by the axis and semi-latus rectum.

12. Of the parabolic segment contained between the curve $y^2=4mx$, and the line $y=ax$.

13. Of the sector of an ellipse, included between the curve and two semi-conjugate diameters.

14. Of the segment of an ellipse cut off by a quadrantal chord.

15. Of the hyperbolic segment contained between the curve $a^2y^2-b^2x^2=a^2b^2$, and the line $x-y=2a$.

16. Of a figure bounded by the arc of a parabola, its directrix, and two lines parallel to the axis.

17. Of the curve $y=\sin x$, between $x=0$ and $x=\pi$.

3. *For a solid of Revolution; the formula is*

$$\bar{x}=\frac{\int xy^2dx}{\int y^2dx},\qquad \text{or } \bar{x}=\frac{\iint r^3\sin\theta\cos\theta\, d\theta dr}{\iint r^2\sin\theta\, d\theta dr}.$$

Find the centre of gravity of the volume—

18. Of a hemisphere.

19. Of the segment of a sphere.

20. Of a semi-prolate spheroid.

21. Of a paraboloid whose axis is a.

22. Of the frustum of a paraboloid; a, b being the radii of its ends, and h the length of its axis.

23. Of a hyperboloid, generated by the curve $y^2=\frac{b^2}{a^2}(2ax+x^2)$.

24. Of the solid formed by the revolution of the sector of a circle about one of its extreme radii.

25. Of the solid generated by the revolution, about the axis of x, of the curve whose equation is $a^2y=ax^2-x^3$.

4. *For any solid; the formulæ are*

$$\bar{x}=\frac{\iiint x dx dy dz}{\iiint dx dy dz},\qquad \bar{y}=\frac{\iiint y dx dy dz}{\iiint dx dy dz},\qquad \bar{z}=\frac{\iiint z dx dy dz}{\iiint dx dy dz}.$$

Find the centre of gravity of the portion—

26. Of a sphere, $x^2+y^2+z^2=a^2$, cut off by three planes
$$x=0,\quad y=0,\quad z=0.$$

27. Of a paraboloid, $y^2+z^2=4ax$, cut off by three planes
$$x=c,\quad y=0,\quad z=0.$$

Ex. 6. Find the centre of gravity of the portion—

28. Of a cylinder, $y^2 = 2ax - x^2$, intercepted between the planes
$$z = cx, \quad z = mx.$$

5. *For a Plane Curve; the formulæ are*
$$\bar{x} = \frac{\int x ds}{\int ds}; \qquad \bar{y} = \frac{\int y ds}{\int ds}.$$

Find the centre of gravity of the arc—

29. Of a semicircle.
30. Of a quadrant of a circle.
31. Of a circle, the length of the arc being s, and of the chord c.
32. Of a semi-cycloid, where $y = a \operatorname{vers}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}$.
33. Of a parabola $y^2 = 4mx$, cut off by the latus rectum.
34. Of the curve, $y = \sin x$; from $x = 0$ to $x = \pi$.

6. *For a Surface of Revolution; the formula is*
$$\bar{x} = \frac{\int xy ds}{\int y ds}.$$

Find the centre of gravity of the surface—

35. Of a segment of a sphere: the equation to the generating circle being $y = (2ax - x^2)^{\frac{1}{2}}$, and c the limiting abscissa.
36. Of a cone whose axis is a.
37. Of a paraboloid whose latus rectum is $4m$ and axis a.
38. Of a figure generated by the revolution of a semi-cycloid,
$$\{y = a \operatorname{vers}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}\} \text{ about its axis.}$$

When the line, area, solid, &c. are of variable density, we must no longer consider the mere volume of an elementary portion, but its mass, which equals the volume × density; and then proceed as before.

Find the centre of gravity—

39. Of a physical line, the density of which at any point varies inversely as the square of its distance from a given point in the line produced.
40. Of the area of a circular quadrant, the density at any point of which varies as the distance from the centre.
41. Of the volume of a right cone, the density of which at any point varies as the square of its distance from a plane through the vertex parallel to the base.
42. Of the volume of a hemisphere, consisting of laminæ parallel to the base; when the density of each lamina varies as the square of its own radius.

Ex. 7.

When a body is suspended by one point, the straight line joining the point of suspension and the centre of gravity is vertical.

1. If a triangle, whose sides are 3, 4 and 5, be suspended from the centre of the inscribed circle; find in what position it will rest.

When a body, acted on by gravity only, is in equilibrium with one point of it resting on a surface, the straight line joining this point of contact with the centre of gravity is vertical.

Therefore, if the body be a solid of revolution, the centre of gravity is always at the intersection of the axis, and the normal at the point of contact.

2. A paraboloid of revolution, whose axis equals a, and radius of base b, rests with its convex surface on a horizontal plane. Find θ the angle of inclination of its axis to the horizon.

3. A paraboloid laid upon a horizontal plane rests with its axis inclined to the horizon at an angle of 30°; compare the length of the axis with the latus rectum.

4. A prolate spheroid rests with its smaller end upon a horizontal plane; determine the nature of the equilibrium.

5. A solid is composed of a cone and a hemisphere, of equal bases, placed base to base. Find the ratio between the dimensions of the cone and hemisphere, in order that the whole may be at rest with *any* point of the spherical surface on a horizontal plane.

Find the centre of gravity of the compound figure, and when this point coincides with the centre of base of hemisphere, the figure will balance as required.

6. A solid is composed of a hemisphere and a paraboloid, of equal bases, placed base to base. Find the ratio between their dimensions, in order that the whole may be at rest with *any* point of the spherical surface on a horizontal plane.

Volume of paraboloid $= \frac{1}{2}$ *(volume of circumscribing cylinder).*

7. A cone rests with its base upon the vertex of a given paraboloid; find the greatest ratio which the height of the cone can bear to the length of the latus rectum of the paraboloid, while the equilibrium remains stable.

8. If two hemispheres rest with the convex surface of one placed on that of the other; determine the nature of the equilibrium.

If a body rest with its base on a plane, it will fall over or not according as the vertical through the centre of gravity falls without or within the base.

9. ABCD is a quadrilateral figure of which the two shorter sides AB, BC are equal, as also the two longer sides, AD, DC; and the angle ABC is a right angle. If the length of AB be given, what is

Ex. 7.

the greatest length of AD, that the figure may rest with the base AB on a horizontal plane, without oversetting?

10. How high can a wall 4 feet thick and inclined at an angle of 75° be built without falling?

11. The slant height of a wall is equal to 3 times its horizontal thickness: determine the inclination when it will be just supported.

12. A cube is placed on an inclined plane whose angle of elevation is 50°. Will it roll or slide?

13. What is the least angle of inclination that a plane must have to the horizon, for a prism on a regular base of n sides to roll down it?

14. A cone is placed with its base on a plane inclined at an angle of 30° to the horizon, and is prevented from sliding. Find the smallest vertical angle it can have that it may not fall over.

15. A paraboloid, of given parameter m, when prevented from sliding down a plane whose inclination is α, just stands on its base without falling over. Find the length of the axis of the paraboloid.

Centre of Parallel Forces.

Ex. 8.

1. Three parallel forces, acting at the angular points A, B, C of a plane triangle, are respectively proportional to the opposite sides a, b, c; find the distance of the centre of parallel forces from A.

2. If three parallel forces, acting at the angular points A, B, C of a given triangle, are to each other as the reciprocals of the opposite sides a, b, c; find the distance of their centre from C.

Guldinus' Properties.

(1) *The surface generated by a curve line, revolving about a fixed axis in its own plane = the product of the length of the curve, into the length of the path described by its centre of gravity.*

(2) *The volume generated by a plane area revolving about a fixed axis in its own plane = the product of the area, into the length of the path described by its centre of gravity.*

Ex. 9.

1. A parallelogram and a triangle, on the same base and between the same parallels, revolve around the base as an axis. Compare the solids they generate.

2. Find the volume generated by the revolution of a right-angled triangle, whose sides are a and b, about the hypothenuse.

3. Determine the volume of the frustum of a right cone, generated by a trapezoid revolving about its altitude h, which joins extremities of the parallel sides a, b.

Ex. 9.

4. Determine the distances of the centres of gravity of a semi-circular area, and a semi-circular arc, from the diameter.

5. Find the surface of a sphere.

6. Find the centre of gravity, of the eighth part of a sphere, or of the solid generated by a quadrant of a circle revolving about one of its radii through an angle of 90°.

7. Find the distance of the centre of gravity of the area of a semi-parabola from its axis.

Volume of a paraboloid $= \frac{1}{2} \times$ *circumscribing cylinder.*

8. Find the surface and volume of the solid generated by the complete revolution of a semi-cycloid about its axis.

9. Find the surface and volume of the solid generated by the revolution of a cycloid about its base.

10. Find the volume of the solid ring generated by the revolution of an ellipse about an external axis in its own plane through an angle of 180°.

11. If any area revolve about an axis, in its own plane, and dividing it into any two parts; show that the *difference* between the solids generated by the parts, will be equal to the whole area multiplied by the path of its centre of gravity.

MACHINES.

Lever.

If a, c, *be the lengths of perpendiculars drawn from the fulcrum upon the directions of the forces* P, W, *respectively; then* $Pa = Wc$.

Ex. 10.

1. A uniform straight lever, 3 feet in length, weighs 4 lb.; what weight on the shorter arm will balance 10 lb. on the longer, the fulcrum being one foot from the end?

2. In rowing, if the oar be 12 feet long, and the rowlock $2\frac{1}{2}$ feet from the handle, compare the pull of the rower with the resistance of the boat.

3. One extremity of a straight lever 20 feet long (without weight) rests on a fulcrum; at what distance from the fulcrum must a weight of 112 lb. be placed, so that it may be supported by a force equivalent to 50 lb. acting at the other extremity?

4. Two weights P, Q are suspended from the extremities of the arms of a straight lever without weight, which are as 3 : 5; P acts at an angle of 60°, and Q at an angle of 45°; find the ratio of P : Q.

5. A bar weighs a ounces per inch. Find its length when a given weight na, suspended at one end, keeps it in equilibrium about a fulcrum at a distance of b inches from the other end.

Ex. 10.

6. Four weights, 1, 3, 7 and 5, are at equal distances on a straight lever without weight. Where must be the fulcrum on which they balance?

7. On a uniform straight lever, weighing 6 lb., and of 6 feet in length, weights of 1, 2, 3, 4, 5 lb. are hung at respective distances of 1, 2, 3, 4, 5 feet from the extremity. Required the position of the fulcrum, about which the whole will rest.

8. If $n+1$ bodies, P, 2P, 3P, 4P, &c. be placed at equal distances along a straight rod without weight, and of length na; find the point on which the whole will balance.

9. If a heavy rod of uniform thickness be moveable about a fulcrum 3 feet from one end A, and 7 feet from the other end B; and a weight of 20 lb. at B be balanced by a weight of 60 lb. at A: required the weight of the rod.

10. A beam, 30 feet long, balances itself on a point at one-third of its length from the thicker end; but when a weight of 10 lb. is suspended from the smaller end, the prop must be moved 2 feet towards it, in order to maintain the equilibrium. Find the weight of the beam.

11. A uniform beam, 18 feet long, rests in equilibrium upon a fulcrum 2 feet from one end; having a weight of 5 lb. at the further, and one of 110 lb. at the nearer end to the fulcrum. Find the weight of the beam.

12. A weight 3W is attached to two strings, whose lengths are AD = 4 ft. BD = 3 ft., at point D; the other ends of the strings being fastened to the extremities of a uniform straight lever of weight W, whose arms AC, BC are 2 and 3 ft. respectively. Find the force which, acting vertically at A, will keep the lever at rest in a horizontal position.

13. AC, CB are the equal arms of a straight lever whose fulcrum is C; to C a heavy arm CD is fixed perpendicular to AB; if now different weights be suspended successively from the extremity A, show that the tangents of the angles, which CD makes with the vertical, will be proportional to the weights respectively.

14. Two weights are suspended from the arms of a bent lever without weight, which are inclined to the horizon at angles of 45° and 30° respectively; the first arm being 18 inches and the second 12 inches long. Find the proportion of the weights.

15. The arms of a bent lever are 3 feet and 5 feet, and inclined to each other at an angle of 150°; and at their extremities weights of 7 lb. and 6 lb. respectively are suspended. Find the inclination of each arm to the horizon, when there is equilibrium.

16. The lengths of the arms, their inclination to each other, and the weight at the extremity of the shorter arm being the same as in the last question; find what the other weight must be, in

Ex. 10.

order that, 1st the shorter, and 2nd the longer arm may rest in a horizontal position.

17. A uniform bent lever, when supported at the angle, rests with the shorter arm horizontal; but if this arm were twice as long, it would rest with the other arm horizontal. Find the ratio between the lengths of the arms; also the angle at which they are inclined to each other.

18. The arms of a bent lever are equal, and the weights suspended at their extremities are as $1 : \sqrt{2}$; find the angle between the arms, that the arm with the less weight may be horizontal.

19. If two forces P and W sustain each other on the arms of a bent lever PCW, and act in directions PA, WA, which form the sides of an isosceles triangle PAW; show that, if AC be joined and produced to meet PW in E, $\quad P : W = WE : PE.$

20. AP, BW are the directions of two parallel forces P, W, which are in equilibrium on the equal arms of the bent lever ACB: draw CD perpendicular to AB, and CE parallel to AP; then

$$P+W : P-W = \tan \tfrac{1}{2}(ACB) : \tan DCE.$$

21. If a lever, kept at rest by weights P, W suspended from its arms a, b, so that they make angles α, β with the horizon, be turned about its fulcrum through an angle 2θ; prove that the vertical spaces described by P and W, are as $a\cos(\alpha+\theta) : b\cos(\beta-\theta)$.

22. A body weighs 10 lb. 9 oz. at one end of a false balance, and 12 lb. 4 oz. at the other end; find the real weight.

23. The whole length of the beam of a false balance is 3 ft. 9 in.; a certain body, placed in one scale, appears to weigh 9 lb.; and placed in the other appears to weigh 4 lb. Find the true weight of the body, and the lengths of the arms of the balance, supposed to be without weight.

24. One pound is weighed at the ends of a false balance, and the sum of the apparent weights is $2\frac{1}{2}$ lb.: what is the ratio of the lengths of the arms?

25. If in a false balance a body weighs p at one end, and q at the other; find the centre of suspension.

26. The same weight is weighed at the two ends of a false balance, and it is observed that the whole gain is $\frac{1}{n}$th part of the true weight: find the distance of the fulcrum from the middle point of the balance.

27. If one of the arms of a false balance be longer than the other by $\frac{1}{m}$th part of the shorter: when used, the weight is put into one scale as often as into the other. What will be the gain or loss per cent. to the seller?

Ex. 10.

28. The beam of a false balance being uniform; show that the arms are respectively proportional to the differences between the true and apparent weights.

29. A Roman steelyard, whose weight is 10 lb., has its centre of gravity at a distance of 2 inches from the fulcrum; and the weight to be determined is supported by a pan placed at a distance of 3 inches on the other side. Find the respective distances from the fulcrum at which the constant weight of 5 lb. must be placed, in order to balance 10, 20, 30, &c. lb. placed successively in the pan.

30. The distance of the centre of gravity of a Danish steelyard from the extremity (where the pan containing the weight to be determined is fixed) is 18 inches; and the weight of the beam is 3 lb. Find at what distances respectively, from the same extremity, the fulcrum must be placed, when weights of 4, 8, 12, 16, 20, 24 ounces are placed successively in the pan.

31. Given the weight Q of a wheel whose radius is 4 feet; find the power, acting in a horizontal direction, required to draw it over a square stone 2 feet in height.

Wheel and Axle.

P : W = *radius of axle* : *radius of wheel.*

Ex. 11.

1. If a power of 10 lb. balance a weight of 555 lb. on a wheel whose diameter is 4 yards; what must be the radius of the axle? The thickness of the rope is here neglected.

2. The radius of the wheel being 2 feet, and of the axle 5 inches, and the thickness of the rope being $\frac{3}{4}$ inch; find what power will balance a weight of 130 lb. The power and weight are supposed to act in the axis of the rope.

3. The radius of a wheel being 3 feet, and of the axle 3 inches; find what weight will be supported by a power of 120 lb. The thickness of the rope coiled round the axle is 1 inch.

4. There are two wheels, whose respective diameters are 5 feet and 4 feet, on the same axle; the diameter of the axle being 20 inches. What weight on the axle would be supported by forces equal to 48 lb. and 50 lb. on the larger and smaller wheels respectively?

5. In a combination of wheels and axles, each of the radii of the wheels, is to each of the radii of the axles, as 5 : 1. If there be 4 wheels and axles, what power will balance a weight of 1875 lb.?

6. In a combination of wheels and axles, in which the circumference of each axle is applied to the circumference of the next wheel, and in which the ratios of the radii of the wheels and axles

Ex. 11.

are 2 : 1, 4 : 1, 8 : 1, &c., there is an equilibrium when the power is to the weight as 1 : p; determine the number of wheels.

7. Two weights of 5 lb. and 7 lb. are attached to two points in the circumference of a wheel, the arc between them being 120°; find the position in which the greatest weight will be supported on the axle.

Pulley.

(1) $W = nP$; n *being the number of strings at the lower block.*
(2) $W = 2^n P$; *the* n *moveable pulleys hanging each by a separate string.*
(3) $W = (2^n - 1)P$; *each of the* n *strings being attached to the weight.*

Ex. 12.

1. At what angle must the strings of a single moveable pulley be inclined to each other, in order that P may equal W?

2. Find the power required to sustain a weight of 100 lb. on a single moveable pulley, when the strings include a right angle.

3. In the system of pulleys where the same string passes round all the pulleys, of which there are 5 at the lower block; find what power will support a weight of 1000 lb.

4. How many pulleys (supposed to be without weight) must there be in the system, where each pulley hangs by a separate string, in order that 1 lb. may support a weight of 128 lb.?

5. In the same system there will be equilibrium, if the power, the weight, and each pulley are all equally heavy.

6. In the same system of 6 moveable pulleys, find the ratio that the weight of each pulley must bear to the power, in order that the latter may just be balanced by the weight of the pulleys alone.

7. In the same system, a weight of 640 lb. is sustained by a power of 5 lb.; what is the number of moveable pulleys?

8. In the same system, n being the number of moveable pulleys, and the strings parallel; if the weights of the pulleys, reckoning from the one nearest to W, increase in a geometric progression, in which 2 is the common ratio; find the relation between P, W, and w the weight of the lowest pulley.

9. In a system of 6 pulleys (supposed to be without weight) where each string is attached to the weight; find what weight will be supported by a power of 12 lb.

10. In the same system of 8 pulleys, what power will be sufficient to sustain a weight of 1020 lb.?

11. In the same system of 8 pulleys, find the ratio that the weight of each pulley must bear to the weight supported, in order that the latter may just be supported by the weight of the pulleys alone.

12. In the same system, n being the number of moveable pulleys

Ex. 12.

and the strings parallel; if the weights of the pulleys, reckoning from the one nearest to P whose weight is w, increase in a geometric progression, in which 2 is the common ratio: find the relation between P, W and w.

13. If, in a system of pulleys where each string is attached to the weight, the weight of the lowest pulley is equal to P, of the second 3P, and so on, that of the highest moveable pulley being $3^{n-2}P$; find the ratio of P : W.

Inclined Plane.

If α *be the inclination of the plane to the horizon,*
ϵ . . . . *power to the plane,*
R . . . *reaction of the plane, or pressure upon it;*
then, $P\cos\epsilon = W\sin\alpha$; $R + P\sin\epsilon = W\cos\alpha$.

Ex. 13.

1. A force of 40 lb. acting parallel to the length sustains a weight of 56 lb. on an inclined plane whose base is 340 feet; find the height and length of the plane.

2. What force is necessary to support a weight of 50 lb. on a plane inclined at an angle of 15° to the horizon; the force acting horizontally?

3. If, on an inclined plane, the pressure, force and weight be as the numbers 4, 5 and 7; find the inclination of the plane to the horizon, and the inclination of the force's direction to the plane.

4. If the weight, power and pressure on an inclined plane be respectively as the numbers 25, 16 and 10; find the inclination of the plane, and the inclination of the power to the plane.

5. A weight W is just supported on an inclined plane by a force P, acting by means of a wheel and axle placed at the top, so that the string attached to the weight is parallel to the plane. Given R and r, the radii of the wheel and of the axle; find the plane's inclination to the horizon.

6. Two planes of equal altitudes are inclined at angles of 60° and 45°, on which P and W are respectively supported by means of a string passing over the common vertex and parallel to the planes; find the ratio P : W.

7. P and W, connected by an inextensible string, balance on two inclined planes which have a common altitude; if the weights receive a small displacement, show that their virtual velocities are inversely as their masses, and that their common centre of gravity has neither ascended nor descended during the motion.

SCREW.

P : W = *dist. between two threads : circumf. described by* P.

Ex. 14.

1. The distance between two contiguous threads of a screw is 2 inches, and the arm at which P acts is 20 inches; determine the ratio P : W.

2. Find the weight that can be sustained by a power of 1 lb., acting at the distance of 3 yards from the axis of the screw; the distance between two contiguous threads being 1 inch.

3. What must be the length of a lever, at whose extremity a force of 1 lb. will support a weight of 1000 lb. on a screw; the distance between two contiguous threads being $\frac{3}{4}$ inch?

4. What force must be exerted to sustain a ton weight on a screw, the thread of which makes 150 turns in the height of 12 inches; the length of the arm being 6 feet?

5. If the thread of the screw be inclined at an angle of 30° to a transverse section of the cylinder whose radius is 9 inches; the length of the lever which turns the screw being 4 feet: find what power will sustain 15 cwt. on it.

FRICTION.

When a body rests on any surface, the friction $=\mu R$; *where* R *is the normal pressure at that point,* μ *being constant and called the coefficient of friction.*

Ex. 15.

1. A given weight W is sustained on a rough plane, whose angle of inclination to the horizon is α, by a power P, inclined at an angle β to the plane; μ being the coefficient of friction. Find between what limits P must lie.

2. Determine the least force which will drag a weight of 50 lb. along a rough horizontal plane, the friction being such as would just prevent the body from sliding down a plane of inclination 30°.

3. An isosceles triangle, whose base is to one of its equal sides as 1 : $7\frac{1}{2}$, is placed with its base on an inclined plane; and it is found that when the body begins to slide, it also begins to roll over. Find the coefficient of friction.

4. A uniform beam, of weight W, leans against a vertical wall, and has its lower end resting on a horizontal plane. If μ and μ' be the coefficients of friction of the wall and of the plane respectively; find the value of θ the inclination of the beam to the horizon, when motion is just on the point of taking place. Find also the pressures on the wall and on the plane.

Resolve vertically, and horizontally, and take moments about the lower end of the beam.

Ex. 15.

5. A ladder rests against a vertical wall, to which it is inclined at an angle of 45°; the coefficients of friction of the wall and of the horizontal plane being respectively $\frac{1}{3}$ and $\frac{1}{2}$; and the centre of gravity of the ladder being at its middle round. A man, whose weight = half the weight of the ladder, ascends it; find to what height he will go before the ladder begins to slide.

Resolve, and take moments as in the last example.

6. A hemisphere rests between a vertical wall and a horizontal plane; μ and μ' being the coefficients of friction of the wall and plane respectively. Find θ the inclination of the base of the hemisphere to the horizontal plane, in the limiting position of equilibrium.

Resolve vertically, and horizontally, and take moments about the centre of the sphere.

7. A uniform and straight plank of length l, rests with its middle point upon a rough horizontal cylinder whose radius is r, their directions being perpendicular to each other; find the greatest weight which can be suspended from one end of the plank whose weight is W, without its sliding off the cylinder.

8. A uniform beam AB, of which the end B presses against a rough vertical plane CD, is supported by a fine string AC attached to a fixed point C in the plane; find the position of the beam when bordering upon motion.

9. An elliptical cylinder, placed between a smooth vertical plane and a rough horizontal one, with the major axis of the ellipse inclined at 45° to the horizon, is just prevented by friction from sliding; required the coefficient of friction.

10. Two equal beams AC, BC are connected by a smooth hinge at C, and are placed in a vertical plane with their lower extremities A and B resting on a rough norizontal plane; if α be the greatest value of the angle ACB for which equilibrium is possible, to determine the coefficient of friction at the ends A and B.

11. A cylinder lies upon two equal cylinders which rest on a horizontal plane, all three in contact and having their axes parallel; find the conditions of equilibrium, that all the points of contact of the cylinders and plane may begin to slip at the same instant.

DYNAMICS.

THE COLLISION OR IMPACT OF BODIES.

Let A, B *be the masses of the two bodies that impinge, one on the other; a, b their velocities before impact, and measured in the same direction; u, v the velocities after impact; e the common elasticity of the two bodies: then will*

$$u=\frac{Aa+Bb-eB(a-b)}{A+B};\qquad v=\frac{Aa+Bb+eA(a-b)}{A+B}.$$

When the bodies are inelastic, $e=0$; *when perfectly elastic,* $e=1$.

Ex. 1.

1. A and B weigh 12 lb. and 7 lb. respectively, and move in the same direction with velocities of 8 feet and 5 feet in a second; find the common velocity after impact; also the velocity lost by A, and that gained by B respectively.

2. A, moving with a velocity of 11 feet, impinges upon B moving in the opposite direction with a velocity of 5 feet, and by the collision A loses one-third of its momentum; what are the relative magnitudes of A and B?

3. A, weighing 8 lb., impinges upon B weighing 5 lb. and moving in A's direction with a velocity of 9 feet in 1 sec.; by collision, B's velocity is trebled; what was A's velocity before impact?

4. A and B are two perfectly hard balls meeting in opposite directions; A is three times as large as B, but moves with a velocity of 12 feet in 1 sec., which is only $\frac{2}{3}$rds of B's velocity; what is the common velocity after impact?

5. A : B :: 3 : 2, and the velocity of A : velocity of B :: 5 : 4; they are perfectly hard bodies, and move before impact in the same direction; find the velocity lost by A, and gained by B.

6. There are 5 inelastic bodies, whose weights are 1, 3, 5, 7, 9 lb.; the first impinges on the second at rest with a velocity of 4 feet in 1 sec.; the second on the third, and so on: find the velocity communicated to the last.

7. A and B are two perfectly elastic balls and in the ratio of 4 : 3; they are moving in the same direction with velocities as 5 : 4; what is the ratio of the velocities of A and B after impact?

8. A and B are perfectly elastic; they are moving in opposite directions; A is treble of B, but B's velocity is double that of A; determine the motions after impact.

9. A and B are two perfectly elastic bodies, and the velocity com-

Ex. 1.

municated from A, to B at rest, is to the velocity retained as 7 : 1; find the ratio of A to B.

10. There is a row of perfectly elastic bodies in an increasing geometrical progression whose common ratio is 3, and placed contiguous to each other; the first impinges upon the second, which transmits its velocity to the third, and so on; the last body moves off with $\frac{1}{64}$th of the velocity of the first body; what was the number of bodies?

11. A perfectly elastic ball 2 lb. in weight, moving with a velocity of 9 feet per sec., communicated a velocity of 2 feet to a body of 8 lb. weight, by means of an intermediate body; find its magnitude.

12. Two perfectly elastic spheres A, B, *meet* directly with equal velocities; find the relation between their magnitudes, that after collision one of them may remain at rest.

13. A, B, and C are three perfectly elastic bodies, and $C=3A$; find B, so that the velocity communicated from A to C through B, may be to the velocity communicated immediately from A to C, as 16 : 15.

14. Determine the ratio of two perfectly elastic balls A, B; so that A, after striking B at rest, may lose $\frac{1}{n}$th of its velocity.

15. A $(=3B)$ impinges upon B at rest; A's velocity after impact is three-fifths of its velocity before impact; find the value of ϵ.

16. The centres of two balls, A and B, move along the same straight line with velocities a and b; find the velocity of each after impact, when $6A=5B$, a is 7 feet per sec., $4a+5b=0$, and $\epsilon=\frac{2}{3}$.

17. Find the elasticity of two spheres A and B, and their proportion to each other, so that when A impinges upon B at rest, A may remain at rest after impact, and B move on with one-sixth part of A's velocity before impact.

18. Two bodies, A and B, whose elasticity is $\frac{2}{3}$, moving in opposite directions with velocities 25 feet and 16 feet per second respectively, impinge directly upon each other; find the distance between them, when $4\frac{1}{2}$ seconds from the moment of impact have elapsed.

19. With what velocity must a ball, whose elasticity is ϵ, impinge upon another equal ball of the same substance moving with a velocity c, in order that the impinging ball may be reduced to rest by the collision?

20. At what angle must a body, whose elasticity is $\frac{1}{3}$, be incident on a perfectly hard plane, that the angle between the directions before impact and after, may be a right angle?

21. Determine the velocities of two bodies A and B, whose elasticity is ϵ, moving in the same direction; so that after collision, A may remain at rest, and B may move along with an assigned velocity.

Ex. 1.

22. A set of five balls, whose elasticity is $\frac{2}{3}$, are in geometric progression, with the common ratio 2; the first impinges on the second at rest, the second on the third at rest, and so on; compare the velocity of the first before impact, with the velocity communicated to the last.

23. The sides AB, CD of a billiard table are parallel; and an imperfectly elastic ball struck from a point C in one side, impinges at E in the other, and is reflected to D in the first side. Show that the time along CE : time along ED $= \epsilon : 1$.

24. An imperfectly elastic sphere, whose elasticity $= \tan 30°$, impinges upon a plane with a velocity such that the velocity after impact $=$ (velocity before impact) $\times \sin 45°$; find the angles of incidence and reflexion.

25. A spherical body A impinges directly with a velocity a upon a spherical body B at rest; their common elasticity being ϵ, find the mass of a third body, which moving with A's velocity before impact, shall have the same momentum which B has after impact.

26. A and B are two balls of given elasticity; what must be the magnitude of a third ball, that the velocity communicated from A to B, by the intervention of this ball, may equal that communicated immediately from A to B? Determine also the limits within which this problem is possible.

27. If A communicate velocity to B through a number of other bodies, which are geometric means between A and B, find the limit to which the velocity of B will continually approach, when the number of means is continually increased, the bodies being perfectly elastic.

28. A ball, whose elasticity is ϵ, projected from a given point in the circumference of a circle, after being reflected from it twice, returns to the given point. Required the direction of projection; and compare the times of describing the first and last chords.

29. A and B are two given points in the diameter of a circle; find in what direction a perfectly elastic body must be projected from A, so that after reflexion at the circle it may strike B.

30. Given the point A between two inclined planes; find the direction of projection, such that the body, after reflexion at each plane, may return to A.

31. A row of four balls A, B, C, D of perfect elasticity is placed in a straight line. Required the ratio of their masses, that the momentum of A may, after impact, be equally divided among the four; B, C, D being originally at rest.

Find the result also when there are n balls.

32. A number of balls A, B, C, ... of given elasticity are placed in a straight line; A, with a given velocity, impinges on B, B then impinges on C, and so on; find the masses of the balls B, C, ..., in order that each of the balls A, B, C, ... may be at rest after im-

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pact on the next; and find the velocity of the nth ball after being struck by the $(n-1)$th.

33. A number n of equal spheres are placed on a smooth table in a straight line and close together: they are connected together by equal inelastic threads of given length l: a velocity a is communicated to the first in the direction of the line passing through their centres, so as to separate it from the second: find the time which elapses before the last sphere is set in motion.

34. One perfectly elastic sphere impinges on another equal sphere at rest, so that the line joining their centres at the impact makes an angle of 45° with the first line of motion; find the angle between the paths of the spheres after impact.

35. Two balls A and 2A, whose elasticity is $\frac{2}{3}$, move with velocities $2a$ and a respectively, and the direction of each makes an angle of 30° with the common tangent at the point of impact; find the directions and velocities of the bodies after impact.

36. A ball A in motion is struck by an equal ball B moving with the same velocity, and in a direction making an angle α with that in which A is moving, in such a manner that the line joining their centres at the time of impact is in the direction of B's motion; find the velocities of the bodies, whose elasticity is ϵ, after impact, and show that that of A will be greatest when $\alpha = 2\cot^{-1}(2-\epsilon)^{\frac{1}{2}}$.

37. An inelastic sphere A moving with a given velocity a impinges upon an inelastic sphere B at rest, the line joining the centres of the two spheres at the instant of collision making an angle α with the direction of A's motion; find the velocity of A after impact.

38. The direction and velocity of the motion of the common centre of gravity of two bodies is not altered by their impact.

39. Two perfectly elastic balls are dropped from two points not in the same vertical line, and strike against a perfectly elastic horizontal plane: show that their common centre of gravity will never re-ascend to its original height, unless the initial heights of the balls be in the ratio of two square numbers.

40. If a perfectly elastic ball be projected from either focus of an ellipse, in any direction within the plane of the figure; show that it will return, after two reflexions from the curve, to the same focus.

41. A, B, C are the weights of three perfectly elastic balls, in the order of their magnitudes; A strikes B at rest with a given velocity, and drives it against C; the distance between B and C being given, find where A will overtake B again.

42. A ball, of given elasticity ϵ, is to be projected in a horizontal plane from a given point P, so that being reflected at any number of given vertical planes in a given order, it may afterwards strike a

Ex. 1.

given point Q; find the direction of projection by a geometrical construction.

Def. The term 'vis viva' *expresses the product of the mass of a body into the square of its velocity.*

43. In the direct impact of two perfectly hard bodies, prove that the difference of the *vires vivæ*, before and after impact, is equal to the sum of the *vires vivæ* of the bodies moving with the velocities lost and gained respectively.

44. Find the sum of the *vires vivæ* of two imperfectly elastic bodies m, m', after direct collision.

45. In the oblique impact of two perfectly elastic bodies, the sum of each body multiplied into the square of its velocity, is the same before and after impact.

46. If A, B, C be three perfectly elastic balls having the velocities a, b, c respectively; and A impinge upon B, and B upon C, so that their velocities after impact are u, v, w respectively, v being the velocity of B after impinging on C: prove that

$$\mathrm{A}a^2+\mathrm{B}b^2+\mathrm{C}c^2=\mathrm{A}u^2+\mathrm{B}v^2+\mathrm{C}w^2.$$

UNIFORMLY ACCELERATED MOTION AND GRAVITY.

Let s denote the space described in the time t seconds by a body acted on by an uniformly accelerating force f. At the beginning of the t sec., the body is supposed to be at rest, and at the end of t sec. to have acquired a velocity v. Then—

1. $s=\frac{1}{2}ft^2=\frac{1}{2}tv=\dfrac{v^2}{2f}.$ 2. $v=ft=\dfrac{2s}{t}=(2fs)^{\frac{1}{2}}.$

3. $t=\dfrac{v}{f}=\dfrac{2s}{v}=\left(\dfrac{2s}{f}\right)^{\frac{1}{2}}.$ 4. $f=\dfrac{v}{t}=\dfrac{v^2}{2s}=\dfrac{2s}{t^2}.$

If a body be projected with a velocity V in a direction coincident with, or opposite to, that in which f acts; then—

5. $s=\mathrm{V}t+\frac{1}{2}ft^2$, or $\mathrm{V}t-\frac{1}{2}ft^2$ *respectively.*

6. $v^2=\mathrm{V}^2+2fs$, or V^2-2fs *respectively.*

If P, Q *be two weights supported on two inclined planes that have a common vertex, and whose angles of inclination to the horizon are α, β respectively; then*

$$7.\ f=\frac{\textit{moving force}}{\textit{mass moved}}=\frac{\mathrm{P}\sin\alpha\sim\mathrm{Q}\sin\beta}{\mathrm{P}+\mathrm{Q}}g;$$

where $g=32{\cdot}19084$ feet$=32{\cdot}2$ feet nearly; $\log g=1{\cdot}507732$.

Ex. 2.

1. A body has been falling for 11 seconds; find the space described, and the velocity acquired.

Ex. 2.

2. Find the time in which a falling body would acquire a velocity of 500 feet; and the height from which it will have fallen.

3. What is the velocity acquired by a body when it strikes the earth, having been dropped from a height of 450 feet; and the momentum acquired by a body of 10 stone weight?

4. A body has been falling for 15 seconds; compare the spaces described in the seventh and last seconds.

5. A body has fallen through a height equal to 440 yards; what was the space described by it in the last second?

6. A body has been falling for $12\frac{1}{2}$ seconds; what was the space described in the last second but 5 of its fall?

7. The space described by a body in the 5th second of its fall was to the space described in the last second but 4, as 1 to 6; what was the whole space described?

8. A body, in falling, has described one-third of the altitude in the last second: determine the altitude, and the time of descent.

9. A body falls from the top of a tower 200 feet high: determine the time of its falling through a part, whose length is two-thirds of its height, and which is so situated that its extremities are equidistant respectively from the top and bottom of the tower.

10. If a body fall through a distance of a feet at two different places, and if the time of falling at one place be T seconds less, and the velocity acquired m feet greater, than at the other; compare the force of gravity at the two places.

11. A body is projected vertically upwards with a velocity of 64 feet per second; how far will it ascend before it begins to return?

12. With what velocity must a stone be projected from the top of a tower, 250 yards above the sea, that it may reach the water in 6 sec.?

13. A stone dropped from a bridge, strikes the water in $2\frac{1}{2}$ seconds; find the height of the bridge. Also, if the stone be projected downwards with a velocity of 3 feet per second; in what time will it strike the water?

14. A body is projected vertically upwards, and the time between its leaving a given point and returning to it is given; find the velocity of projection, and the whole time of motion.

15. Upon a steeple 150 feet high is a spire of 40 feet; at the same instant that a stone was let fall from the top of the steeple, another was projected vertically upwards from the bottom of it, with a velocity sufficient to carry it to the top of the spire; at what point will these stones meet?

16. A body is projected upwards from the lower extremity of a vertical line, 250 feet high, with a velocity of 90 feet per second; after what time must another be projected downwards from the upper extremity with the same velocity, so as to meet the former in the middle point of the line?

Ex. 2.

17. A body projected vertically upwards from the bottom of a tower, with a velocity equal to that acquired through 1·8 times its height, rose to the top in 2 seconds; find the height of the tower.

18. A body projected in the direction of the action of a constant force describes P and Q feet in the *p*th and *q*th seconds; find the magnitude of the force, and the velocity of projection.

19. A body is dropped from a height of 400 feet; after it has fallen through 50 feet, another is projected from the same height, with such a velocity that the two bodies reach the ground at the same time: find the velocity of projection.

20. If, from the extremities of a vertical line h, two bodies be projected at the same time, one downwards with a velocity a, the other upwards with a velocity c; determine where they will meet.

21. With what velocity must a body be projected downwards, that in n sec. it may overtake another body, which has already fallen a feet?

22. A body, whose elasticity is $\frac{1}{2}$, projected from the floor of a room 12 feet high, strikes the ceiling and floor, and just reaches the ceiling again; find the velocity of projection.

23. A rocket, ascending vertically with a velocity of 100 feet in 1 sec., explodes when it has reached its greatest height, and the interval between the sound of the explosion reaching the place of starting, and a place $\frac{1}{4}$ mile distant, is 1 sec.; determine the velocity of sound.

24. An imperfectly elastic body is projected downwards with a given velocity against a hard horizontal plane, and being reflected, just reaches the point of projection in t sec.; find the distance of the plane from this point, and the elasticity of the body.

25. An imperfectly elastic ball is projected downwards with a certain velocity, and after falling through a height of 20 feet meets a horizontal plane; the ball rebounds 10 feet, then falls again and rebounds 4 feet: find the elasticity, and velocity of projection.

26. A ball of given elasticity is projected vertically upwards with a velocity of 40 feet per second; it returns to the point of projection, which is on a hard horizontal plane, and rebounds; it returns again and rebounds, and so on, till the motion ceases; find the whole space described.

27. A ball, whose elasticity is $\frac{3}{5}$, falls from a height of 50 feet upon a hard horizontal plane, and rebounds continually till its velocity is destroyed: find the whole space described.

28. Two balls A, B perfectly elastic, are dropped at the same instant from two given points in the same vertical line; find the point where B, after rebounding from the horizontal plane, will meet A.

29. Two perfectly elastic balls, beginning to descend from different

Ex. 2.

points in the same vertical line, impinge upon a perfectly hard plane inclined at an angle of 45°, and move along a horizontal plane with the velocities acquired; find what distance they will move along the horizontal plane before collision takes place.

30. PQ is a given vertical line terminating in a hard horizontal plane at Q; a perfectly elastic ball being dropped from P meets another perfectly elastic ball rebounding with a known velocity from Q, and both are reflected back: to find where they must meet in order that they may thus rebound from one another continually.

Ex. 3.

1. A body falls 9 feet along an inclined plane in the first second; find the inclination of the plane.

2. The length of an inclined plane is 400 feet, its height 250; a body falls from rest from the top of the plane; what space will it have fallen through in $3\frac{1}{2}$ seconds; what time will it be in falling through 300 feet; and what velocity will it have acquired when it has arrived within 50 feet of the bottom of the plane?

3. The elevation of a plane is 25° 30′; a body, in falling from the top to the bottom of it, acquires a velocity of 450 feet per second; required the length of the plane.

4. A body descending vertically draws an equal body 25 feet in $2\frac{1}{2}$ seconds up a plane inclined at 30° to the horizon, by means of a string passing over a pulley at the top of the plane; determine the force of gravity.

5. Two bodies start from the top of an inclined plane, one falling down the length of the plane, and the other down its height; it is observed that the former is 3 times as long as the latter in reaching the base. Required the inclination of the plane.

6. The length of an inclined plane is 40 feet, and its inclination is 30°; mark out upon it a part, equal to the height, through which a body, falling down the plane, will move whilst another body would descend freely through the height.

7. Divide the length of a given inclined plane into three parts, so that the times of descent down them successively may be equal.

8. If a body be projected down a plane inclined at 30° to the horizon, with a velocity equal to three-fourths of that due to the height of the plane, the time down the plane will equal the time down its vertical height, from rest.

9. A given weight P draws another given weight W up an inclined plane of given height and length, by means of a string parallel to the plane; when and where must P cease to act, that W may just reach the top?

10. A body falls down a given inclined plane, and at the instant when it begins to fall, another is projected upwards from the bottom

Ex. 3.

of the plane with a velocity equal to that acquired in falling down an equally inclined plane n times its length; where will they meet?

11. A body is projected from the top of a given inclined plane with a velocity a, and after the lapse of n sec., another is projected from the bottom with a velocity c; both bodies moving along the plane, determine where they will meet.

12. A body of given elasticity ϵ is projected up an inclined plane with a given velocity, and at the top impinges perpendicularly upon another plane, and returns to the point from which it set out with two-thirds of the velocity of projection; find the length of the plane, and the limits of ϵ.

13. If a be the base of an inclined plane; determine its height, so that the time of a body's falling down the plane may be the least possible.

14. Determine that point in the hypothenuse of a right-angled triangle whose base is parallel to the horizon, from which the time of a body's descent to the right angle may be the least possible.

15. A right-angled triangle being placed with its two sides horizontal and vertical respectively, find their ratio, so that the time of falling down the vertical and describing the base with the velocity acquired, may be equal to the time of descent down the hypothenuse.

16. AC BC are two inclined planes meeting a horizontal plane at the same point C; having given the inclinations of the planes and the length of BC, find the point A, such that an inelastic body descending down AC may just ascend to the top of BC.

17. Two equilateral triangles in the same vertical plane are placed with their bases at a distance of d feet from each other upon the same horizontal line, and a non-elastic body falls down the side of the first, moves along the space between the bases and up the side of the second triangle, the vertex of which it just reaches; given the side of the first triangle equal to a feet; find the side of the second, and the whole time of motion.

18. If two heavy particles begin to fall at the same time from the common vertex of two inclined planes, the line joining them will move parallel to itself.

19. Through what chord of a circle, drawn from the extremity of a vertical diameter, must a body fall, to acquire half the velocity it would acquire in falling down that diameter?

20. Determine that diameter of a circle down the last half of which a body descends in the same time as down the whole vertical diameter.

21. In a vertical circle, two chords are drawn from the extremity of a horizontal radius subtending arcs θ and 2θ: if the time down the chord of 2θ equal n times that down the chord of θ; show that $\sec\theta = n^2 - 1$.

Ex. 3.

22. Two bodies, A and B, descend from the same extremity of the vertical diameter of a circle, A down the diameter, and B down the chord of 30°. Find the ratio of A to B, when their centre of gravity moves along the chord of 120°.

23. In an inverted parabola, the time of descending down any chord from a point P to the lowest point, is equal to the time of descending vertically to a horizontal line, which is at a distance below the vertex equal to the latus rectum.

24. If from any point in a rectangular hyperbola, whose axis is vertical, two lines be drawn to the extremities of this axis; the times of descent down them will be equal.

25. In a hyperbola whose major axis is horizontal, determine the diameters down which a heavy body will descend in a given time; also that diameter down which it will descend in the shortest time.

26. The plane of a cycloid, whose axis is a, being inclined to the horizon at an angle of 60°; find the time of descent down a chord, drawn from the vertex to either extremity of the base.

27. AB is a vertical line of given length: find the locus of a point P, such that the square of the time down AP, plus the square of the time down PB, starting from rest at P, may be constant.

28. Two bodies fall from two given points in the same vertical, down two straight lines drawn to any point of a curve, in the same time; all the lines are in the same vertical plane; find the equation to the curve.

29. Two balls, whose weights are 9 lb. and 2 lb. respectively, are connected by a string 15 feet in length; the 9 lb. is supported on a smooth horizontal table, along which it is drawn by the 2 lb. that begins to fall; find the velocities acquired by the latter after falling vertically through 12 feet, and 20 feet; also the times of motion in both cases.

In the latter case, let the string be supposed to break when the 9 lb. reaches the edge of the table.

30. Two equal weights W are suspended over a fixed pulley; what weight must be added to one of them, that it may descend through 100 feet in 8 seconds?

31. A mass of 18 lb. is so distributed at the extremities of a cord passing over a fixed pulley, that the more loaded end descends through 13 yards in as many seconds; required the weights at each end.

32. A weight of 7 lb. draws up one of 5 lb. over a fixed pulley; at the instant of letting go the weight of 7 lb. a velocity 4 feet is communicated to it; how far will it descend in 8 seconds, and what velocity will it have acquired at the end of that time?

33. If P, Q be two bodies connected by a string passing over a

Ex. 3.

smooth fixed pulley and P descend; after P has described a given space a, let a weight p be removed from P, leaving the remainder $P-p$ less than Q. Determine the subsequent motion.

34. A weight P after falling freely through a feet, begins to raise up a weight $Q>P$, connected together by means of a string passing over a smooth fixed pulley; find the extreme height to which Q can rise, and the time of its ascending.

35. Find the straight line of quickest descent from a given point within a circle to the circumference.

36. Find the straight line of quickest descent, from the focus to the curve of a parabola, whose axis is vertical.

37. The major axis of an ellipse is vertical; determine the radius vector measured from the upper focus, down which the time of descent is the least possible.

MOTION UPON A CURVE, AND THE SIMPLE PENDULUM.

If l=number of inches in the length of a simple pendulum,
t sec.=time of one oscillation;

$$t=\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}.$$

If, in the latitude of London, L *be the length of the seconds' pendulum, and g the force of gravity;*

$L=39{\cdot}1393$ *in.*; $g=32{\cdot}19084$ *ft.*; $\pi=3{\cdot}14159$
log $L=1{\cdot}592613$; *log* $g=1{\cdot}507732$; *log* $\pi={\cdot}497150$.

Ex. 4.

1. Three planes A, B, C are in contact; A is vertical, B and C are inclined to the horizon at angles 60° and 30° respectively; find the velocity, with which a body beginning to descend from A will begin to move along the horizontal plane passing through the lower extremity of the plane C.

2. A ball having descended to the lowest point of a circle through an arc whose chord is a, drives an equal ball up an arc whose chord is b; find the common elasticity e of the two balls.

3. If θ be the angular distance of a body from the lowest point in a circular arc; show that the force in the direction of the arc is to the force in the direction of the chord as $2\cos\frac{1}{2}\theta : 1$.

4. Having given the length of the seconds' pendulum, find the length of a pendulum that will oscillate 4 times in one second; and another 9 times in one minute.

5. In what time would a pendulum, 80 inches long, vibrate at the distance of two of the earth's radii, above the surface of the earth?

Ex. 4.

6. Find the length of a pendulum which oscillates as often in one minute as there are inches in its length.

7. Find the length of a pendulum that would oscillate three times, whilst a heavy particle falls from rest through 81 feet.

8. If a pendulum vibrate seconds at the earth, it would vibrate minutes at the moon, the distance of the moon from the earth's centre being taken equal to 30 times the earth's diameter.

9. A seconds' pendulum is lengthened 1·05 inches; find the number of seconds it will lose in 12 hours.

10. A pendulum which should beat seconds, is found to lose 20 seconds a day. Determine the quantity by which its length should be increased or diminished.

11. A pendulum gains 3 seconds in an hour, before it is carried up a high mountain; at what height in the ascent would the pendulum keep true time, if the earth's radius were 4000 miles?

12. How high must a seconds' pendulum be carried above the level of the sea, that it may vibrate 598 times in 10 minutes, the radius of the earth being 3958 miles?

13. A seconds' pendulum is carried to the top of a mountain and there loses 48·6 seconds in a day; determine the height of the mountain, supposing the earth's radius to be 4000 miles.

14. The length of a pendulum that vibrates sidereal seconds being 38·926 inches; find the length of a sidereal day. Find the increment of the length of the pendulum, that it may measure mean solar time.

15. A pendulum, which would oscillate seconds at the equator, would, if carried to the pole, gain 5 minutes a day; compare the forces of equatoreal and polar gravity.

16. Two pendulums, the lengths of which are L and l, begin to oscillate together, and are again coincident after n oscillations of L. Given L the greater, to find l.

17. Two pendulums, A and B, begin to oscillate together, and are again coincident after 12 oscillations of A; find the length of B, that of A being 38·3 inches and longer than B.

18. A pendulum, which vibrates seconds at Greenwich, taken to another place is found to lose n seconds a day; compare the forces of gravity at the two places.

19. If a clock, at a place A on the earth's surface, keeps true time, and when taken to another place B loses n minutes daily, but goes right on being shortened by the mth part of an inch; find the length of the pendulum.

20. A pendulum 40 inches long oscillates 3·5 times between the time of seeing the flash and hearing the report of a cannon; find the distance of the cannon from the observer, the velocity of sound being assumed equal to $35g$ per second.

21. If from the extremity of the vertical diameter of a circle the

Ex. 4.

chord of 60° be drawn; compare the time of falling down this chord with the time of oscillation of a pendulum equal in length to the chord.

In the interior of the earth—
Gravity varies directly as the distance from the earth's centre.

22. A seconds' pendulum, on being carried to the bottom of a mine, is found to lose 10 seconds a day; determine the depth of the mine, if the earth's radius be 4000 miles.

23. How far below the earth's surface, or how high above it, must the pendulum, whose length is 39·12 inches, be taken to oscillate seconds, the earth's radius being 3958 miles?

24. If a pendulum, when carried to the top of a mountain, is observed to lose in a given time just twice as much as it does when taken to the bottom of a mine in the neighbourhood; show that the height of the one is equal to the depth of the other.

25. The times of oscillation of a pendulum are observed at the earth's surface, and at a given depth below the surface; hence determine the radius of the earth, supposed spherical.

26. A particle, acted on by gravity, descends from any point in the arc of an inverted cycloid, of which the axis is vertical, to the lowest point of the curve; find the time of descent.

PROJECTILES IN A NON-RESISTING MEDIUM.

If two straight lines be drawn through the point of projection, one horizontal, the other vertical; and these lines be taken for the coordinate axes of x and y respectively; V *the velocity of projection, and α the angle which the direction of projection makes with the axis of x; h the space due to the velocity of projection; then the equation to the curve described by the projectile is*

$$1.\quad y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha}.$$

If R *be the horizontal range,* T *the time of describing this range,* H *the greatest height, then*

$$2.\quad T=\frac{2V}{g}\sin\alpha. \qquad 3.\quad R=2h\sin 2\alpha. \qquad 4.\quad H=h\sin^2\alpha.$$

If R′, T′ *be the range and time respectively on an inclined plane passing through the point of projection, and having an elevation ι; then*

$$5.\quad R'=4h\frac{\sin(\alpha-\iota)\cos\alpha}{\cos^2\iota}. \qquad 6.\quad T'=\frac{2V}{g}\,\frac{\sin(\alpha-\iota)}{\cos\iota}.$$

If w be the weight of a ball or shell, p the weight of the gunpowder

used in firing the ball or shell from a mortar, and v the velocity generated by the powder, then

$$7.\ v=1600\left(\frac{3p}{w}\right)^{\frac{1}{2}} \text{ feet.}$$

Ex. 5.

1. A body is projected in a direction making an angle of 15° with the horizon, with a velocity of 60 feet per second; find its range, greatest altitude, and time of flight.

2. A body is projected at an angle of 45°, and descends to the horizon at a distance of 500 feet from the point of projection; with what velocity was it projected, what was its greatest altitude, and the whole time of flight?

3. The horizontal range of a projectile is 1000 feet and the time of flight is 15 seconds; find the direction and velocity of projection; also the greatest altitude of the body during the flight.

4. The horizontal range of a body, projected at an angle of 15°, is 841 feet; find how high the body would rise, if projected vertically upwards with the same velocity of projection.

5. If the horizontal range of a projectile be to the greatest height as $4 : 3\frac{1}{2}$; find the angle of projection.

6. If the horizontal range of a body, projected with a given velocity, be three times the greatest altitude, find the angle of projection. Find this angle when the range is equal to the altitude.

7. Find the velocity and direction of projection of a ball, that it may be 100 feet above the ground at the distance of half a mile and may strike the ground at the distance of 1200 yards.

8. From one extremity of the base (500 feet) of an isosceles triangle, whose vertical angle is 36°, situate in a vertical plane, a body is projected in the direction of the side adjacent to that extremity, so as to strike a body placed in the other extremity; find the velocity of projection, and the time of flight.

9. A shell being discharged at an angle of 45°, its explosion was heard at the mortar $3\frac{1}{2}$ seconds after the discharge; required the horizontal range, the velocity of sound being $35g$ per second.

10. A ball is fired at a given elevation α towards a person who is on the same horizontal plane as the gun; if the ball and the sound of the discharge reach him at the same instant, find the range; the velocity of sound being $35g$ per second.

11. If a body be projected with a velocity of 850 feet per second in a direction making an angle of 60° with the horizon; find the focus of the parabola described, and also its latus rectum.

12. Find the velocity and direction, with which a body must be projected from a given point, that it may hit two other given points in the same vertical plane.

13. If the area of the parabola described by a projectile be one-

Ex. 5.

third of the square described on the horizontal range; find the angle of projection.

14. Two bodies are projected from the same point with the same velocity; the directions of projection are measured by the angles α and 2α respectively; compare the areas of the parabolas described, supposing the horizontal ranges equal.

15. If the areas in the last problem be equal; what is the value of α?

16. Two bodies are projected from the same point with equal velocities, and their horizontal ranges are equal; if the areas of the parabolas described are as $2:1$; find the angles of projection.

17. A shot is fired with a given velocity towards a tower whose horizontal distance from the cannon is one-half the range, and whose altitude subtends an angle $\tan^{-1}\frac{3}{4}$ at the point of projection; find the inclination of the cannon to the horizon, that the shot may strike the summit of the tower.

18. At a distance a from the bottom of a vertical line, a ball is projected at an angle of $45°$, which just touches the top, and afterwards strikes the ground at the distance b from the bottom on the other side; find the height of the line.

19. A body is projected at an angle of $60°$ elevation, with a velocity of 150 feet per second; find the direction and velocity of the projectile after the lapse of 5 seconds; also its height above the horizontal plane passing through the point of projection.

20. Two bodies are projected from the same point with equal velocities so as to describe the same horizontal range, and the times of flight are as $3^{\frac{1}{2}}:1$; required the directions of projection.

21. If two bodies be projected from the same point with equal velocities, and in such directions that they both strike the same point on a plane whose inclination to the horizon is β; if α be the angle of projection of the first, compare the times of flight.

22. AB is the vertical diameter of a circle. A perfectly elastic ball descends down the chord AC, and being reflected by the plane BC, describes its path as a projectile; show that the body will strike the circle at the opposite extremity of the diameter CD.

23. A ball of given elasticity is projected with a given velocity at a given elevation. On meeting the horizontal plane it rebounds, describes another parabola, and again rebounds; and so describes a series of parabolæ. Find the whole horizontal distance described before the ball ceases to rebound.

24. The velocities at the extremities of any chord of the parabola described by a projectile, when resolved in a direction perpendicular to the chord, are equal.

25. A body projected from the top of a tower, at an elevation of $30°$ above the horizon, fell in n seconds at a distance of a feet from the base; find the height of the tower.

Ex. 5.

26. With what velocity must a body be projected from a tower, in a direction parallel to the horizon, so that it shall strike the ground at a distance from the foot of the tower equal to half its height?

27. A body is projected, from the top of a tower, 200 feet high, with a velocity of 50 feet per second and at an angle of elevation of 60°; find the range on the horizontal plane passing through the foot of the tower, and the time of flight.

28. A body is projected from the summit of a hill, whose form is a right cone the vertical angle of which is 120°, in a given direction with a given velocity; to find where the projectile will strike the hill.

29. A body is projected from the summit of a mountain of 45° elevation, so as just to strike at the bottom, and with double the velocity of projection, which equals that due to a height of 400 yards; find the height of the mountain and the greatest height attained by the projectile.

30. A body, projected in a direction making an angle of 30° with a plane whose inclination to the horizon is 45°, fell upon the plane at the distance of 250 feet from the point of projection, which is also in the inclined plane; required the velocity of projection, and the time of flight.

31. At the foot of a tower 60 feet high runs a river 300 feet in width; a hill slopes from the opposite bank of the river at an angle of 30° to the horizon; from the top of the tower a cannon-ball is fired at an elevation of 60°; the impetus is 500 feet: at what distance from the bank of the river will the ball strike the hill?

32. The heights of the ridge and eaves of a house are 40 feet and 32 feet respectively, and the roof is inclined at 30° to the horizon. Find where a sphere, falling down the roof from the ridge, will strike the ground, and also the time of descent from the eaves.

33. If a body be projected up a plane AC, inclined at 45° to the horizon, with the velocity acquired in falling down a vertical line $=AC$; find the range AD on the horizontal plane passing through the point A. Determine also the time between its leaving the point of projection and meeting the horizontal plane.

34. Given the base of an inclined plane; find its inclination so that a body projected directly up it, with a given velocity, may after passing the top, fall at the greatest possible distance beyond the base in the same horizontal line.

35. A body is projected with a velocity of 160 feet per second, and at an angle of 45° with the horizon; after the lapse of 5 seconds, an object, dislodged by the projectile, strikes the ground; required the distance of the object struck from the point of projection.

36. From the top of a tower, two bodies are projected with the

Ex. 5.

same velocity at different given angles of elevation, and they strike the horizontal plane at the same place; find the height of the tower.

37. Two bodies A and B are projected at the same instant, from the same point, with velocities u and v respectively—the one vertically and the other at an elevation of $30°$; find the path described by their common centre of gravity.

38. Four balls, whose weights are 2, 3, 5 and 6 pounds respectively, are projected from the same point at the same instant and with the same velocity of 1000 feet per second; the angles of elevation are severally $22° 30'$, $30°$, $45°$ and $60°$; find the height of their common centre of gravity after 3 seconds have elapsed, and the highest point to which it will rise.

39. A body is projected vertically upwards from a point A, with a given velocity; find the direction in which another body must be projected with a given velocity from a point B in the same horizontal line with A, so as to strike the first body.

40. Several bodies being projected in different directions from the same point and with the same velocity; it is required to find the locus of all the bodies at the end of a given time.

41. Find the locus of the vertices of all the parabolas described under the circumstances of the last problem.

42. Planes AP, AP′, AP″, &c. being drawn in every direction from the point A, and bodies projected from A with a given velocity at such angles that the ranges on each of these planes shall be the greatest; to find the locus of all the extreme points P, P′, P″, &c.

43. A body of given elasticity slides down an inclined plane of given length, whose inclination is $\cos^{-1}\frac{1}{\sqrt{3}}$, and impinges on the horizontal plane at the foot of the inclined plane; required the range of the body after reflection at the horizontal plane.

44. An imperfectly elastic ball, from a given height is let fall on a given inclined plane; required the point at which it will again strike the plane after reflection.

45. A perfectly elastic ball falls from a height h on a plane inclined at an angle of $30°$ to the horizon; after what time will the ball again strike the plane; and what is the distance between the two points struck?

46. A perfectly elastic body is projected from a point in a plane whose inclination to the horizon is ι; find the angle of projection in order that after striking the plane the body may be reflected vertically upwards.

47. Two balls are projected at the same instant from two given points in a horizontal plane and in opposite directions so as to describe the same parabola. What must be their relative magnitude

Ex. 5.

and elasticity, so that after impact one of them may return through the same path as before and the other descend in a right line?

48. The time of describing any portion PQ of the parabolic path of a projectile is proportional to the difference of the tangents of the angles, which the linear tangents at P and Q make with the horizon.

49. If two particles be projected from the same point, at the same instant, with velocities u, v, and in directions α, β respectively; find the time which elapses between their transits through the other point, which is common to both their paths.

50. An imperfectly elastic ball is projected with a given velocity and in a given direction; when the ball is at its greatest height it is reflected directly by a vertical plane; determine where the ball will strike the horizontal plane, passing through the point of projection, and the whole time of flight.

51. If u, v, w be the velocities at three points P, Q, R of the path of a projectile, where the inclinations to the horizon are α, $\alpha-\beta$, $\alpha-2\beta$, and if t, t' be the times of describing PQ, QR respectively: show that $wt=vt'$, and $\frac{1}{u}+\frac{1}{w}=\frac{2\cos\beta}{v}$.

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52. A ball of 13 lb. weight is fired from a mortar on the summit of a mountain with a charge of 6·5 oz. of powder, so as just to strike at the bottom and with double the velocity of projection; find the height of the mountain.

53. How much powder is required to throw an 8-inch shell 1500 yards on an inclined plane passing through the point of projection, its inclination being $28°\,45'$, and that of the mortar $48°\,30'$?*

54. A gun is mounted on a citadel 450 feet above the level of the sea. A pirate is observed in a line, making an angle of depression $=5°\,30'$, and it is required to fire into her a 13-inch shell: how much powder will be necessary, and what is the time of flight, the inclination of the gun being $24°\,30'$?*

ROTATION OF BODIES.

I. Moment of Inertia.

If k be the radius of gyration of any system of particles m, m', m'', &c. capable of moving about an axis at the distances r, r', r'', &c. respectively, then

$$k=\left(\frac{mr^2+m'r'^2+m''r''^2+\&c.}{m+m'+m''+\&c.}\right)^{\frac{1}{2}}=\left\{\frac{\Sigma(mr^2)}{\Sigma(m)}\right\}^{\frac{1}{2}}, \quad or \quad \left\{\frac{\int r^2 dm}{\int dm}\right\}^{\frac{1}{2}}.$$

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* *The 8-inch and 13-inch shells are supposed to weigh 48 lb. and 196 lb. respectively.*

Ex. 6. Find the radius of gyration—

1. Of a right line or slender rod about an axis through its extremity and perpendicular to its length.

2. Of a plane circle or wheel revolving on its centre.

3. Of a circular arc about a radius through its vertex.

4. Of a circular arc about an axis perpendicular to its plane and passing through its centre of gravity.

5. Of a circular arc about an axis perpendicular to its plane and passing through its vertex.

6. Of a circular area revolving about a straight line parallel to its plane, at a distance c from its centre.

7. Of an elliptic area about its principal axes.

8. Of an elliptic area about an axis through the centre perpendicular to its plane.

9. Of an isosceles triangle about a perpendicular let fall from its vertex upon its base.

10. Of a triangular lamina about an axis perpendicular to its plane and passing through one of its angular points.

11. Of a triangular lamina about an axis through its centre of gravity and perpendicular to its plane.

12. Of a parallelogram about an axis through its centre of gravity and perpendicular to its plane.

13. Of a regular polygon of n sides, about an axis through the centre and perpendicular to its plane.

14. Of an annulus about a perpendicular axis through the centre.

15. Of a parabolic area bounded by the curve and a double ordinate to the axis, about a line through the vertex perpendicular to the plane.

16. Of a sphere about a diameter.

17. Of a spherical shell about a diameter.

18. Of a cylinder about its axis.

19. Of a solid cylinder about an axis, passing through the middle point of, and perpendicular to, its own axis.

20. Of a right cone about its axis.

21. Of a right cone about an axis passing through its vertex and perpendicular to its own axis.

22. Of a paraboloid about its axis.

23. If the density of a straight rod AB vary as the nth power of the distance from one end A, and k, k' be the radii of gyration of the rod about axes at right angles to its length through A, B respectively; compare the values of k, k'; also find n, when $k=6k'$.

24. The moment of inertia of any plane figure about any axis, equally inclined to the principal axes, which have the same origin, is equal to two-thirds of the greatest of the moments about those principal axes.

II. Centre of Oscillation.

If l, h be the distances of the centres of oscillation and gravity of a system of particles from the axis of suspension, then

$$l = \frac{mr^2 + m'r'^2 + m''r''^2 + \&c.}{(m+m'+m''+\&c.)h} = \frac{\Sigma(mr^2)}{h.\Sigma(m)}, \text{ or } \frac{\int r^2 dm}{h.\int dm}.$$

Ex. 7. Find the time of oscillation—

1. Of an isosceles triangle, about an axis through its vertex perpendicular to its plane.

2. Of three equal weights placed at the angles of an equilateral triangle without weight, which is suspended by an axis, perpendicular to its plane and bisecting one of its sides.

3. Of a regular hexagon, about an axis through one of its angular points, and perpendicular to its plane.

4. Of a circular arc of 60°, about an axis through its middle point perpendicular to its plane.

5. Of a cube, about one of its edges.

6. Of a square pyramid, about an axis through its vertex perpendicular to its geometrical axis.

7. Of a solid cylinder, about a line in its surface parallel to the axis.

8. Of a sphere, about an axis touching its surface.

9. Of a right cone, about an axis which is a tangent to the circumference of its base.

10. Of a paraboloid, about an axis through its vertex perpendicular to its geometrical axis.

11. Of the solid, generated by a sector of 60° of a given circle, revolving about one of its extreme radii; the solid being suspended from the vertex.

12. Two equal heavy particles are fixed, one at the middle point, and the other at the extremity of a rigid imponderable rod, and suspended from the other extremity; find the time of an oscillation.

13. A heavy particle is suspended at a distance of 30 inches from a horizontal axis; at what distance must another particle of double the weight be placed, so that the two, rigidly connected, may together vibrate seconds?

14. If three pendulums, consisting of three equal heavy particles attached to the same horizontal axis by rigid imponderable rods, of lengths, such that their times of oscillation are 1 sec., 2 sec., 3 sec. respectively; find the time of oscillation, when all three are rigidly connected together.

15. A pendulum consists of a rigid imponderable rod AA′, 52 inches long, and 2 spheres of radii 5 and 3 inches, placed with their centres at A, A′ respectively; find the time of oscillation about a horizontal axis through a point S in the rod, such that A′S equals 12 inches.

Ex. 7.

16. A rigid rod SA without weight, 5 feet long, passes through the centres of three spheres C, B, A whose radii are 2, 3, 5 inches respectively, so that SC=40 in., CB=12 in., BA=8 in., S being the point of suspension; find the time of an oscillation.

17. Find at what point of the rod of a perfect pendulum, must be fixed a given weight of indefinitely small volume so that the pendulum may vibrate in the shortest time possible.

18. If the interior of two circles, which touch internally, be taken away, and the remaining area oscillate about an axis in its own plane which is a tangent at the point common to the two circles; find the centre of oscillation.

19. Compare the times in which a circular plate will vibrate about a horizontal tangent, and about a horizontal axis through the point of contact at right angles to the tangent.

20. Find the isosceles triangle of a given area, which, vibrating about an axis passing through its vertex perpendicular to its plane, shall oscillate in the least time possible.

21. A sector of a circle revolves about an axis perpendicular to its plane, and passing through the centre of the circle; find the angle of the sector when the length of the isochronous simple pendulum equals one-half the length of the arc.

22. Prove that a right cone, whether suspended at its vertex or by the diameter of its base, will oscillate in equal times; the height of cone being equal to the radius of its base.

23. Determine the ratio of the diameter of the base to the altitude of a cone, so that the centre of oscillation, when the cone is suspended by the vertex, may be in the centre of the base.

24. Find the dimensions of a cone of given volume, which, being suspended by the vertex, will oscillate as many times in a minute as there are inches in the length of its axis.

25. A cylindrical rod of given length oscillates seconds, when suspended from one extremity; at what point must it be suspended to oscillate once in n seconds?

26. A pendulum consists of a rigid rod OA without weight, and a sphere of which the centre is A and radius r; to determine the point A′ in the line OA at which the centre of another sphere of radius r' must be fixed in order that the time of oscillation of the system of 2 spheres may be the least possible.

27. Two straight rods, equal in length, are suspended by their extremities, one being of uniform density, and the density of the other varying as the nth power of the distance from the axis of suspension; the times of their small oscillations are found to be as $5^{\frac{1}{2}} : 6^{\frac{1}{2}}$; required the value of n.

28. A bent lever, whose arms are a, b, and inclination to one another θ, makes small oscillations in its own plane about the angular point; find the centre of oscillation.

Ex. 7.

29. A uniform rod of length a is bent into the form of a cycloid, and oscillates about a horizontal line joining its extremities; find the length of the isochronous pendulum.

30. If two particles whose weights are as 2 : 3 oscillate, one in a semicircle, and the other in a cycloid; show that the whole tensions of the two strings, at any given inclination of them to the horizon, are equal; the motion in both cases beginning from the highest point.

III. D'Alembert's Principle.

Ex. 8.

1. Two heavy particles P, P′ are attached to a rigid imponderable rod APP′, which is oscillating in a vertical plane about a fixed point in its extremity A; determine the motion.

2. Two particles, attached to the extremities of a fine inextensible thread, are placed upon two inclined planes having a common vertex; determine the motion of the particles and the tension of the thread at any time.

3. One body m draws up another m' on the wheel and axle; determine the motion of the weights and the tensions of the strings.

4. A body P, draws another body Q, over a fixed pulley AB; determine the motion, and tensions of the string.

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5. A square is capable of revolving about a side; find where it must be struck perpendicularly that there may be no impulse upon the axis.

6. Suppose a cylinder, that weighs 100 lb., to revolve upon a horizontal axis, and to be set in motion by a weight P of 15 lb. attached to a string which is coiled round the surface of the cylinder; find the space through which the weight descends in 5 seconds.

7. If P in the preceding Ex. be 20 lb. and descend through 75 feet in 3 seconds; what is the weight of the cylinder?

8. A sphere C, of radius 3 feet and weight 500 lb., is put in motion by a weight P of 20 lb. by means of a string going over a wheel whose radius is 6 inches; in what time will P descend through 50 feet, and what velocity will it then have acquired?

9. A paraboloid whose weight W is 200 lb. and radius of base 20 inches, is made to revolve about its axis, which is horizontal, by means of a weight P of 15 lb. acting by a cord that passes over a wheel of one foot diameter on the same axis; after P has descended for 10 seconds, it is removed, and the paraboloid is left to revolve uniformly with the velocity acquired; find the velocity of the centre of gyration of the paraboloid, and the number of revolutions it will perform in one minute.

Ex. 8.

10. Two weights of 5 lb. and 3 lb. hang over a fixed pulley, whose weight is 12 oz.; find the time of either weight moving through a space of 30 feet.

11. What weight could be raised through a space of 30 feet in 6 seconds by a weight of 50 lb. acting by means of a string going round a fixed and a moveable pulley, the weight of each pulley being 1 lb.?

12. A weight of 500 lb. is raised by a rope wound round an axle whose radius is 6 inches; the weight of the wheel and axle is 80 lb., and the distance of their centre of gyration from the axis of rotation is 3 feet; find the radius and weight of the wheel, so that another weight of 100 lb., acting at its circumference, may make the 500 lb. ascend through a space of 10 feet in 5 sec.; also find the pressure upon the axis during the motion.

13. A hemisphere oscillates about a horizontal axis, which coincides with a diameter of the base; if the base be at first vertical, find the ratio of the greatest pressure on the axis to the weight of the hemisphere.

HYDROSTATICS.

PRESSURE ON SURFACES.

I. *Let* A *be the area of any surface immersed in a fluid,* $\bar{x}$ *the depth of the centre of gravity of the surface* A *below the surface of the fluid,* ρ *the density of the fluid, and* P *the* normal *pressure on* A; *then*

$$P = A\bar{x}\rho.$$

II. *The* vertical *pressure on* A *is equal to the weight of the fluid incumbent on* A.

Ex. 1.

1. An equilateral triangle is immersed in a fluid so that one side is vertical; compare the pressures on the three sides.

2. Two equal isosceles triangles are just immersed vertically in a fluid, one with its base, the other with its vertex downwards; find the ratio of the pressures.

3. An isosceles triangle with its base downwards is just immersed vertically in a fluid; divide the triangle by a line parallel to the base so that the pressures on the upper and lower parts may be as 1 : 7.

4. A triangle, the area of which is A, being immersed in a fluid with its angular points at depths h, k, l below the surface of the fluid; it is required to find the pressure on the triangle.

5. Two equal squares are just immersed vertically in a fluid, one with a side, the other with a diagonal vertical; find the ratio of the pressures.

6. Two squares, whose sides are 9 and 5 inches respectively, are immersed vertically in a fluid, their sides being parallel to its surface. The first square has its upper side at a depth of 4 inches beneath the surface; find the depth to which the second square must be sunk, so that the pressure on it may be 3 times that on the first.

7. The sides of a rectangle immersed vertically in a fluid are 9 and 14, the shorter side being coincident with the surface. From one of the angles at the surface draw a straight line to the base, dividing the rectangle into two parts, such that the pressures on them may be in the ratio of 5 : 3.

8. A rectangle, whose sides are 20 and 7, is immersed vertically in a fluid, with its shorter side coincident with the surface; divide the rectangle into 5 parts by horizontal lines, so that the pressures on each part may be equal.

9. A parallelogram, of which the diagonals AC, BD intersect in

Ex. 1.

O, is immersed in a fluid, so that AB is in the surface of the fluid; compare the pressures L, M, N on the triangles AOB, BOC, COD.

10. A circle whose radius is 6, is just immersed vertically in a fluid; find the radius of a circle touching the former internally at the surface of the fluid, so that the pressures on the smaller circle and frustum may be as 5 : 4.

11. If a circle be inscribed in a square, and another square be inscribed in the circle, and the whole figure be then just immersed vertically in a fluid, so that an angular point of the greater square may coincide with the surface; compare the pressures on the squares and circle.

12. A rectangle is described about a parabola, and the whole figure is immersed vertically in a fluid, so that the vertex coincides with the surface of the fluid; compare the pressures on the parabola and rectangle.

13. A parabola is immersed in a fluid with its axis vertical, and vertex coincident with the surface; divide the parabola by a horizontal line into two parts, so that the pressures on them may be as $m : n$.

14. A parabola with its axis vertical, has its vertex coincident with the surface of the fluid in which it is immersed; divide the parabola by horizontal lines into four parts, so that the pressures on them may be equal.

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15. If a cubical vessel be filled with fluid and rest on one of its sides; compare the vertical and lateral pressures.

16. A cubical vessel filled with fluid is held with one of its diagonals vertical; compare the pressures on the sides.

17. If a cubical vessel be filled, half with mercury and half with water; compare the pressure on the sides with the pressure on the base, which is horizontal.

18. A side of the base of a square pyramid is 10 inches, the altitude is 22 inches; if the pyramid be filled with water, compare the pressure on the base with the pressure on each side, and with the weight of the water.

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19. If two spheres whose radii are as 3 : 5, be just immersed in a fluid; compare the pressures on them.

20. If a given sphere be just immersed in a fluid; compare the pressure on the sphere with the weight of the fluid displaced.

21. If the density of mercury be 13·568 times that of water; it is required to compare the pressure on the internal surface of a sphere filled with water, with the weight of a sphere of mercury of the same radius.

Ex. 1.

22. Compare the pressures on the upper and lower halves of a hemispherical basin filled with fluid.

23. If a hollow spherical segment 5 inches in height be cut from a sphere whose diameter is 16, and be filled with water; determine the lateral pressure.

If N *represent the normal pressure,* M *the vertical pressure, and* L *the lateral pressure, then*

$$N^2 = M^2 + L^2.$$

24. Divide a hollow sphere filled with fluid, by a circle parallel to the horizon, into two parts, so that the pressures on them may be equal.

25. A hemisphere is immersed in a fluid, with its base coincident with the surface of the fluid; divide the hemisphere by horizontal planes into four parts, so that the pressures on all the convex surfaces may be equal.

26. Compare the whole pressure on the surface of a spherical segment filled with fluid, with the weight of the fluid.

27. The radii of the ends of a frustum cut from a sphere, whose diameter is $2r$, are a, b, and the height is h: find the depth to which this frustum must be sunk in a fluid with its axis vertical, that the pressures on its two ends may be equal to n times the pressure on its curve surface.

28. A solid hemisphere is immersed in a fluid with its axis inclined at an angle α to the vertical, the surface of the fluid being a tangent plane to the hemisphere; find the whole pressure on the convex surface of the solid.

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29. The height of a cylindrical vessel filled with fluid is equal to the diameter of the base; compare the pressures on the base and concave surface, with the weight of the fluid.

30. A solid cylinder is immersed in a fluid, the depths of the centres of its circular ends being h, k, its radius r and length d; determine the pressure on the whole surface.

31. A solid cylinder is just immersed in a fluid with its axis vertical; divide the cylinder by a horizontal plane into two parts, such that the pressures on the convex surfaces may be equal.

32. A cylindrical yard-stick is just immersed vertically in a fluid; divide it into three parts which shall sustain equal pressures.

33. A cylinder filled with fluid with its axis vertical is divided by horizontal sections into 5 annuli, so that the pressure on each annulus may be equal to the pressure on the base; the radius of the cylinder is 7 inches; determine its height and the breadth of the third annulus.

Ex. 1.

34. Divide a given cylinder which is just immersed in a fluid with its axis vertical, into n parts so that the pressures on them may be equal.

35. A cylinder 15 inches in length is just immersed vertically in two fluids that do not mix, whose densities are as 1 : 2; and the pressures upon the two parts of the convex surface of the cylinder are as 2 : 3; find the length of the cylinder immersed in each fluid.

36. Divide a cylinder whose axis is 20 inches, into four parts, such that when the cylinder is just immersed vertically in a fluid, the pressures may be in Geometrical Progression with 2 for the common ratio.

37. A cylinder, the length of whose axis is $2h$, rests with its axis vertical in two fluids that do not mix, whose densities are as 1 : 2; the distance between the surfaces of the fluids is h; find the density of the cylinder that it may be just immersed.

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38. Find the whole pressure on the surface of a solid cone including its base, when immersed in a fluid with its axis vertical and its vertex just at the surface of the fluid.

39. If a right cone, its axis being vertical, is just immersed in a fluid, (1) with its base, (2) with its vertex, downwards; compare the pressures on its convex surface in each case.

40. A right cone with its axis vertical is just immersed in a fluid, vertex downwards; divide the cone by a horizontal plane, so that the pressures on the convex surfaces above and below the section may be equal.

41. What is the least depth of fluid, whose density is ρ, in which a cone, whose height is h and density σ, can rest with its axis vertical, the vertex of the cone touching the base of the vessel?

42. What must be the vertical angle of a conical vessel, so that when placed with its vertex upwards and filled with heavy fluid through a hole at the vertex, the pressure on the concave surface may be to the pressure on the base as 4 : 3? Show that the ratio of these pressures cannot for any cone be less than 2 : 3.

43. A given paraboloid filled with fluid stands upon its base which is horizontal; compare the pressure on the concave surface with the weight of the fluid.

44. Determine the form of a vessel of revolution, which being filled with fluid, is such that the pressures on all horizontal sections are equal to one another.

The Differential or Integral Calculus, or both, may be required in **Ex. 2.**

Ex. 2.

1. A circle is just immersed vertically in a fluid; divide the circle

Ex. 2.

by a horizontal line into two parts, so that the pressures on them may be equal.

2. Find the pressure on a loop of the Lemniscata of Bernouilli, whose equation is $r^2 = a^2 \cos 2\theta$, the loop being just immersed with its axis vertical.

3. If a spherical vessel be filled with fluid, determine that horizontal section which sustains the greatest pressure; and compare that pressure with the pressures on the two surfaces into which the sphere is thus divided.

4. A hemisphere, with a flat lid and filled with fluid, is held with a point in its edge uppermost; to find its position when the sum of the pressures on the concave and plane surfaces is the greatest possible.

5. If an inverted paraboloid with its axis a vertical, be filled with fluid; find the depth of that horizontal section on which the pressure is greatest.

Normal Pressure of Heterogeneous Incompressible Fluids.

6. Compare the pressures on a plane figure immersed in a fluid at different depths, but always horizontal; when the density of the fluid varies as the nth power of the distance from its surface.

7. A semicircle is immersed vertically in a fluid with its diameter coincident with the surface; determine on which of the chords parallel to the surface the pressure is greatest, when the density of the fluid increases as the square of the depth.

8. Find the pressure on a triangular plane, one angle of which is a right angle, and the base of which coincides with the surface of the fluid: the inclination of the plane to the horizon is α, and the density of the fluid varies as the depth.

9. A given parabola is just immersed in a fluid with its axis vertical: determine the double ordinate, on which the pressure is greatest, when the density of the fluid increases directly as the depth.

10. A circular area is just immersed, vertically in a fluid, the density of which increases directly as the depth; determine the whole pressure on the area.

11. A cycloidal area is immersed in a fluid, whose density varies directly as the depth, the axis being vertical and the vertex at the surface; determine the whole pressure on the area.

12. Determine the magnitude of a sphere of given density, which will rest just immersed in a fluid, whose density varies as its depth.

13. A cylinder, having its axis vertical, is filled with fluid, the density of which varies directly as the depth; find the whole pressure on the concave surface. Also, find the pressure on the portion

Ex. 2.

of the concave surface, included between two horizontal sections, at depths d, d'.

14. A conical vessel is filled with fluid whose density varies as the depth; if the pressure on the base, which is horizontal, equal the pressure on the concave surface; find the vertical angle.

15. A given cone, with its vertex downwards, is filled with fluid, the density of which varies as the square of the depth; determine the horizontal section, on which the pressure is greatest.

16. A right cone resting with its base on a horizontal plane is filled with fluid, the density of which varies as the nth power of the depth; compare the pressure on the concave surface with the weight of the fluid.

17. An oblate spheroid, generated by the ellipse $9y^2+x^2=36$, the major axis being vertical, is filled with fluid, the density of which varies as the depth; find the horizontal section which sustains the greatest pressure, and the value of that pressure.

CENTRE OF PRESSURE

Of a plane Surface immersed in any homogeneous incompressible Fluid.

Let the plane of the immersed area, inclined at any angle to the horizon, be produced to meet the surface of the fluid, and let the line of intersection be taken for the axis of y: draw the axis of x in that plane perpendicular to the axis of y,

Let $m_1, m_2, m_3 \ldots$ be elemental portions of the immersed area,

$x_1y_1, x_2y_2, x_3y_3 \ldots$ their coordinates respectively,

$\bar{x}\ \bar{y}$ the coordinates of the centre of pressure; then

$$\bar{x}=\frac{m_1x_1^2+m_2x_2^2+m_3x_3^2+\ldots}{m_1x_1+m_2x_2+m_3x_3+\ldots}=\frac{\Sigma(mx^2)}{\Sigma(mx)} \text{ or } \frac{\iint x^2dxdy}{\iint xdxdy}$$

$$\bar{y}=\frac{m_1x_1y_1+m_2x_2y_2+m_3x_3y_3+\ldots}{m_1x_1+m_2x_2+m_3x_3+\ldots}=\frac{\Sigma(mxy)}{\Sigma(mx)} \text{ or } \frac{\iint xydxdy}{\iint xdxdy}$$

the limits of integration being determined by the extent and form of the area.

If the immersed area be symmetrical with regard to the axis of x, then $\bar{y}=0$.

Ex. 3.

1. Find the centre of pressure of a parallelogram immersed in a fluid, one side of the area being in the surface of the fluid.

2. Find the centre of pressure of a rectangular plank immersed vertically to any depth within a fluid, the ends of the plank being horizontal, and at depths h, k.

3. If a square whose side is 20 inches, be immersed vertically

Ex. 5.

7. If the weight of a globe in air be W, and in water w; determine its diameter and Sp. Gr.; having given ρ, α as the Sp. Grs. of water and air respectively.

8. If a body weighs 6 lb. in air, and 2 lb. in water; and another body weighs 7 lb. in air and 4 lb. in water; compare their Sp. Grs.; that of air being ·001225, and of water unity.

9. A diamond ring weighs $69\frac{1}{2}$ grains, and when weighed in water $64\frac{1}{2}$; if the Sp. Grs. of gold and diamond be 16·5 and 3·5 respectively, find the weight of the diamond.

10. The crown of Hiero, with an equal weight of gold and an equal weight of silver were all weighed in water. The crown was found to lose $\frac{1}{14}$, the gold $\frac{4}{77}$, and the silver $\frac{2}{21}$, of the common weight. In what proportion were the gold and silver mixed in the crown?

11. The Sp. Grs. of pure gold and copper being 19·3 and 8·8; find the Sp. Gr. of standard gold, which is an alloy of gold and copper in the proportion of 11 : 1.

12. Given the Sp. Grs. of lead $=11·352$, of cork $=·24$, of white fir $=·569$; how much cork must be added to 56 lb. of lead that the united mass may weigh as much as an equal bulk of fir?

13. Find the weight of a hydrometer, which sinks as deep in rectified spirits of Sp. Gr. ·866, as it sinks in water when loaded with 60 grains.

14. The weight of a piece of wood in air is 4 lb., of a piece of lead in water 4 lb., and of the lead and wood together in water 3 lb.; required the Sp. Gr. of the wood, that of water being 1, and of air ·001225.

15. A body weighs 4 oz. in a vacuum, and if another body which weighs 3 oz. in water be attached to it, the united mass in water weighs $2\frac{1}{4}$ oz.; find the Sp. Gr. of the former body.

16. If 21 pints of sulphuric acid of Sp. Gr. 1·84 be mixed with 8 pints of water, it is found that the mixture measures only 28 pints; find its Sp. Gr.

17. A pint of rectified spirits of Sp. Gr. ·866, is added to a pint of water; if the mixture measure 2 pints, what is its Sp. Gr.?

18. If, in a mixture of two fluids whose Sp. Grs. are 3, 5 respectively, a body whose Sp. Gr. is 8, loses half its weight; compare the quantities mixed.

19. Three fluids whose Sp. Grs. are 4, 5, 7 are mixed together in the proportion of 5, 7, 9 volumes respectively; find the Sp. Gr. of the compound.

20. If P, Q, R be the apparent weights of the same substance in three fluids, whose Sp. Grs. are λ, μ, ν respectively; show that $P(\nu-\mu)+Q(\lambda-\nu)+R(\mu-\lambda)=0$.

21. Two fluids, whose volumes are v, v', and Sp. Grs. ρ, ρ' re-

Ex. 5.

spectively, on being mixed, are found to have lost in volume $\frac{1}{n}$th part of the sum of their original volumes; find the Sp. Gr. of the mixture.

22. A cone and a paraboloid of the same altitude, floating in a fluid with their vertices downwards, have each one-sixth of the axis above the surface of the fluid; compare their Sp. Grs.

23. A sphere of 10 inches diameter floats between two fluids, naphtha and water, whose Sp. Grs. are ·708 and 1 respectively; if five-eighths of the surface of sphere be immersed in the water, find the Sp. Gr. of the sphere.

24. A hollow copper sphere, whose internal diameter is 2 feet, just floats in water; find its thickness, when the Sp. Gr. of the copper is 8·788.

25. Three equal globes, whose Sp. Grs. are 3, 4, 6, are placed in the same straight line. How must they be disposed, so that they may balance on the same point of the line *in vacuo* and in water?

26. A cone whose axis is vertical floats between two fluids whose Sp. Grs. are ρ, ρ' and sinks to the depth of two-thirds of its axis in the heavier (ρ'); find the Sp. Gr. of cone.

27. In a cylinder three-fourths filled with water, a hydrometer is observed to sink to a certain depth; if the cylinder be now filled up with a fluid of 3 times the Sp. Gr. of water, find the weight with which the hydrometer must be loaded to make it sink to the same depth in the mixture.

28. A body weighs 10 lb. *in vacuo* and another 5 lb. in water; their volumes are 48 and 72 cubic inches respectively: compare their Sp. Grs., assuming that a cubic foot of water weighs 1000 oz.

29. A ship on sailing into a river sinks 2 inches, and after discharging 12000 lb. of her cargo rises 1 inch; determine the weight of the ship and cargo, the Sp. Gr. of sea-water being 1·026.

30. Find the depth to which a sphere, of diameter 10 inches and Sp. Gr. ·75, will sink in water.

31. Find the depth, to which an inverted paraboloid, made of wood Sp. Gr. ·65, whose axis is 18 inches, will sink in water.

32. If a cylinder, placed in a fluid with its axis vertical, rest with two-thirds immersed; and when placed in another fluid rest with four-fifths immersed; determine to what depth it would sink in a mixture composed of equal volumes of these fluids.

33. A cylinder, placed with its axis vertical in a fluid (ρ), rests with an mth part immersed; when placed in another fluid (ρ') it rests with an nth part immersed; find the depth to which the cylinder would sink in a mixture composed of a parts of the first fluid, and b parts of the second.

34. A spherical bubble, composed of matter whose Sp. Gr. is σ,

Ex. 3.
in a fluid, with a side horizontal, to such a depth that the distance of the centre of pressure from the centre of gravity is $1\frac{2}{3}$ inches; determine the depth.

4. Find the centre of pressure of a square, whose side is a, immersed to a depth h below the surface, one of its diagonals being vertical.

5. Find the centre of pressure of a triangle, of which one side a, coincides with, and another b, is perpendicular to, the fluid surface.

6. Find the centre of pressure of a trapezoid, whose parallel sides are a and b, the former being coincident with the surface of the fluid, and h the distance between the parallel sides.

7. A circular area is just immersed in a fluid, find the centre of pressure.

8. Find the centre of pressure of a semicircle, having its diameter vertical, and the upper extremity of it in the fluid surface.

9. Find the centre of pressure of a quadrant, having one of its bounding radii coincident with the fluid surface, and the other vertical.

10. Find the centre of pressure of the sector of a circle, whose centre is in the surface of the fluid, and axis vertical.

11. Find the centre of pressure of a parabola, the directrix of which is coincident with the surface of the fluid.

12. A hollow cube filled with fluid is held with one of its diagonals vertical; find the centre of pressure on one of its lower faces.

13. The staves of a cylindrical tub full of water are to be kept together by a single hoop; find where it must be placed.

14. If a flood-gate move upon a vertical axis, the area on one side of the axis being the quadrant of a circle, and on the other side a rectangle of the same altitude; determine the width of the rectangle so that the gate may just open by the pressure of the water when it has risen to the top.

15. If a sphere filled with water, be divided by a vertical plane into two hemispheres; determine the position and magnitude of the lateral forces which shall just prevent their separation.

16. The axis of a cylindrical vessel, containing a given quantity of fluid, is inclined at a given angle to the horizon; determine the centre of pressure of its base.

Heterogeneous Incompressible Fluids.

17. If a rectangular board whose sides are a, b, be immersed vertically in a fluid, the density of which varies as the square of the depth, and the side a coincide with the surface, find the centre of pressure.

18. A semicircular area is just immersed vertically in a fluid, the

Ex. 3.

density of which varies as the depth, with its diameter coinciding with the surface; find the centre of pressure.

19. A parabolic area, cut off by a double ordinate to the axis h, is immersed vertically in a fluid, the density of which varies as the depth; if the tangent at the vertex lies in the surface of the fluid, find the depth of the centre of pressure.

Equilibrium of two fluids of different densities in a bent tube.

If two fluids that do not mix, meet in a bent tube, the altitudes of their surfaces above the horizontal plane in which they meet, are inversely as their densities.

Ex. 4.

1. If equal lengths of two fluids whose densities are as $n : 1$ be poured into a circular tube; determine their position when at rest, n being > 1.

2. If α, β be the angles subtended at the centre, by the lengths of a circular tube, occupied respectively by two fluids that do not mix, and whose densities are ρ, σ, the former being the heavier; find their position of equilibrium.

3. If equal lengths of two fluids whose densities are ρ, ρ' be poured into a cycloidal tube; determine the altitudes of their upper and common surfaces.

SPECIFIC GRAVITY.

The weight of a solid, immersed in a fluid, is equal to the difference *between the true or absolute weights of the solid, and of an equal volume of the fluid.*

Ex. 5.

1. A cube, each edge of which is 4 inches long, weighs 16247 grains in air, and 95 grains in water; determine the weight of a cubic inch of water, if the Sp. Gr. of water $= 770 \times$ Sp. Gr. of air.

2. The edges of two cubes are as $3 : 5$, and their Sp. Grs. as $3 : 4$; find the ratio of their weights.

3. The weights of two globes are as $11 : 3$, and their Sp. Grs. as $3 : 2$; compare their diameters.

4. If a cubic inch of metal weigh 10·36 oz., and a cubic foot of another metal 960 lb. avoirdupois; compare their Sp. Grs.

5. A globe and cylinder, whose surface would circumscribe the globe, are formed of different substances, but are of the same weight; compare their Sp. Grs.

6. A uniform rod of iron, $8\frac{1}{2}$ inches long, and of Sp. Gr. 7·2, floats vertically in mercury, whose Sp. Gr. is 13·568; find the length of the portion immersed.

Ex. 5.

and filled with gas of Sp. Gr. ρ, just floats in air of Sp. Gr. α: determine the thickness of the bubble.

35. If the Sp. Gr. of iron, alcohol and water be 7·2, ·8, and 1 respectively, find the internal radius of a spherical shell of iron $\frac{1}{10}$ inch thick, which, when filled with alcohol, will just float in water.

36. If a hemispherical vessel, of which the weight is W, float upon a fluid with one-third of its axis below the surface; find the weight which must be put into the vessel, so that it may float with two-thirds of its axis below the surface.

37. A cylinder of Sp. Gr. ·5, has its axis and diameter of base, each $=9$ inches; compare the depths to which the cylinder, its axis being vertical, will sink in water, when under an exhausted receiver, and when the air (Sp. Gr. ·001225) has been admitted.

38. A sphere sinks in water, under an exhausted receiver, to a depth $=$·65 of its diameter; find the depth of immersion after the air (Sp. Gr. ·001225) has been admitted.

39. If a wooden ball, connected by a small wire with a ball of lead of the same given radius, be dropped into the sea, and upon striking the bottom, the wooden ball be disengaged and rise to the surface; the whole time elapsed and the Sp. Grs. of the balls and sea-water being given, find the depth of the sea.

EQUILIBRIUM OF FLOATING BODIES.

Ex. 6.

1. A triangular prism floats with its axis horizontal and one edge immersed; find its metacentre.

2. In the oblique position of equilibrium of an isosceles triangle floating in a fluid, show that the circle, described through the vertex and extremities of the line of floatation, will bisect the base.

3. If a triangular prism, the sides of whose base are 5, 6, 7, float in a fluid with its smallest angle immersed; and if the surface of the fluid divide the longest side so that three-fifths of it are immersed; find the position of the prism.

4. If a square of Sp. Gr. ρ be immersed in water; show that if one angle be immersed, there will be 12 different positions of equilibrium, if ρ lie between $\frac{8}{32}$ and $\frac{9}{32}$; and if three angles be immersed, there will be 12 different positions, if ρ lie between $\frac{23}{32}$ and $\frac{24}{32}$.

5. A right-angled triangular board, floats in a fluid with its right angle immersed and hypothenuse horizontal; find its metacentre.

6. Find the density of a square lamina, which floats in a given fluid, with one angle in the surface and two below it.

7. A right cone, whose vertical angle is 90°, floats with its vertex

Ex. 6.

downwards in a fluid whose Sp. Gr. is to that of the cone as $8:1$; determine the nature of the equilibrium.

8. A hollow right cone, whose vertical angle is 60°, is placed with its axis vertical and vertex downwards; find what quantity of fluid it must contain, in order that a given sphere, whose Sp. Gr. is half that of the fluid, may sink just deep enough to touch the surface of the cone.

9. A solid cone of uniform density, whose axis equals the radius of its base, floats with its axis vertical and vertex downwards in two fluids which do not mix; the axis is trisected in the points in which it meets the surfaces of the two fluids; if the densities of the fluids be as $3:2$, determine the nature of the equilibrium.

10. Find the metacentre of a right cone, floating in a fluid with its axis vertical and vertex downwards.

11. What must be the vertical angle of a cone, which floats with its vertex downwards, in order that the metacentre may be at the centre of the plane of floatation?

12. If a solid cylinder of Sp. Gr. ·5, having an elliptic base, be placed in water; determine the nature of the equilibrium.

13. A cone of given Sp. Gr., rests in a given fluid with its vertex immersed and axis vertical; show that the nature of the equilibrium will not be affected by altering the altitude of the cone; and find the vertical angle when the equilibrium is neutral.

14. Find the least density of a cone, whose vertical angle is 90°, which can float in stable equilibrium, with its vertex downwards, in a given fluid.

15. Find the depth to which a given paraboloid must be immersed, with its axis vertical and vertex downwards, in a fluid of three times the density of the paraboloid, in order that it may remain there at rest.

16. Find the metacentre of a paraboloid generated by the parabola $y^2 = 10x$, having its Sp. Gr. $= \cdot 75$ and floating in water with its axis vertical and vertex downwards.

17. A rectangular board floats vertically in a fluid with two of its sides horizontal; find the measure of its stability.

18. A paraboloid, with its axis vertical and vertex downwards, floats in a fluid with half its axis immersed; compare the axis of the paraboloid with the latus rectum of the generating parabola, when the solid is in a position of indifferent equilibrium.

ELASTIC FLUIDS.

The pressure of the atmosphere at the earth's surface, it is assumed, is measured by a column of 33 feet of water, or 30 inches of mercury.

Ex. 7.

1. Show that the height of a homogeneous atmosphere is the same for all elevations above the earth's surface.

2. If a cylindrical vessel 20 feet long be half-filled with water and then inverted, the open end communicating with a basin of water; find the altitude at which the water will stand in the cylinder.

3. A cylindrical tube, 24 inches long and closed at one end, contains mercury which occupies 16 inches of its length; if the tube be now inverted and the open end be inserted into a basin of mercury; find the altitude of the mercury when at rest.

4. The nth part of a given cylindrical tube being filled with air; determine the quantity of mercury to be poured in at the top so as just to fill the tube.

5. A cylindrical tube, 37 inches long, and closed at one end, is filled with $21\frac{2}{3}$ inches of mercury, with 14 inches of water, and with air in the remaining part; find the depth to which the mercury would subside after immersing the open end of the tube in a basin of mercury.

6. A cylindrical tube, 4 feet in length, closed at its upper end, is let down in a vertical position into the sea; having observed that the water had risen in the tube 3 feet, find the depth to which it had been sunk; assuming that a column of 32 feet of sea-water measures the weight of the atmosphere.

7. If an inverted hemisphere full of air be forced down, so as just to be immersed in mercury; construct the equation, from which x, the height to which the mercury rises in the hemisphere, is to be determined.

8. If a hollow paraboloid, mouth downwards, whose axis is 10 feet long, be sunk vertically till the water rises internally 5 feet; determine the distance of the vertex from the surface of the water.

9. If a paraboloid of given dimensions be immersed with its axis vertical and mouth downwards in water to a depth h of the vertex, and the water rise in it to a height k; find the density of the air in the vessel at first.

10. A piston, weighing 5 lb., closely fitting a vertical tube full of common air, whose length is 2 feet, diameter 3 inches and closed at the bottom, descends by its own weight; find the distance of the piston from the top of the cylinder when it has ceased to descend.

A cubic foot of mercury, it is assumed, weighs 13568 oz.

11. Two vertical cylindrical tubes, of given diameters and alti-

Ex. 7.

tudes, one of which is hermetically sealed, and the other open at the top, are connected by a third which is horizontal and filled with water, so that the air in the sealed branch may be in its natural state: a column of water of the same base and altitude as the open tube being poured in, determine the space through which it will descend in that branch.

If z be the height in feet above the earth's surface, t, T the temperatures, indicated by Fahrenheit's thermometer, of the air, and mercury in the barometer whose altitude is h, at the lower station; t', T$'$, h' the corresponding values at the upper station, then will

$$z=60345\left(1+\frac{t+t'-64}{900}\right)\log\frac{h}{h'\{1+(\mathrm{T}-\mathrm{T}')\cdot0001\}}.$$

If the temperatures at the two stations be considered each 32°, this formula becomes

$$z=60345\log\frac{h}{h'}.$$

12. According to Gen. Roy (Phil. Trans. 1777) the mean height of the mercury in a barometer on Carnarvon Quay was observed to be 30·151 inches, the temperature of the mercury 59°·9 Fahr., and that of the air 59°·9. On the top of Snowdon the height of the mercury was 26·474 inches, the temperature of the mercury 50°·88, that of the air 49°·1. Find the height of Snowdon above Carnarvon Quay.

13. Near the summit of Chimborazo, the barometer was observed by Humboldt to fall to 14·85 inches, the attached thermometer being then 50° Fahr. and the detached 29°·12. The same barometer, carried down to the shore of the Pacific Ocean, rose to 30 inches, while both the attached and detached thermometers stood at 77°·54. What was the height attained by the traveller?

14. If a spherical balloon, of which r is the radius, be filled with gas whose Sp. Gr. is α, that of air being 1, and be loaded with a weight W; determine how high it will rise.

15. When the balloon of the last question floats in the air, and a given weight w of ballast is thrown out; find the additional height to which the balloon will rise, and how much a barometer in it will sink, the temperature being considered uniform.

16. Find the radius of a spherical balloon, filled with gas whose Sp. Gr. is to that of air as 1 : 10, the whole weight of balloon with its appendages being 800 lb., that it may just rise 2 miles high; supposing one cubic foot of air at the earth's surface to weigh 1·2 oz., where its density is 4 times as great as at the height of 7 miles.

17. If the internal radius of a spherical balloon made of copper $\frac{1}{100}$ inch thick, be 50 feet, and the balloon be filled with gas of Sp.

Ex. 7.

Gr. $\cdot 1$, that of air being unity; how high will it rise in the atmosphere, the weight of the car, &c. being 5000 lb.?

A cubic foot of air weighs 1·225 *oz., of copper* 8788 *oz.*

INSTRUMENTS AND MACHINES.

Ex. 8.

1. If the radii of the cistern and tube be 3 and 1 in the common barometer; determine the true variation corresponding to an apparent rise or fall of one inch.

2. Find the length of a water-barometer inclined at 60° to the horizon, corresponding to 31 inches of a mercurial barometer, the Sp. Gr. of mercury being 13·568.

3. Some air being left in a barometer tube 33 inches long, it is found that the mercury in it stands at 29 inches when in a perfect barometer it is at 30; find the altitude in the imperfect instrument, when that in the perfect is 25 inches.

4. If, when the mercury in a true barometer stands at an altitude h, the mercury in an imperfect barometer of length l stands at the altitude a; find the altitude in the true barometer corresponding to an altitude c in the imperfect instrument.

5. Two barometers, of the same length l, being imperfectly filled with mercury, are observed to stand at the heights a, a' on one day, and b, b' on another. Determine the quantity of air left in each, the temperature being constant.

6. The weights of a body in air are a, a' corresponding to the heights h, h' of the barometer; find the weight corresponding to a height h''.

If R, b *be the capacities of the Receiver and barrel of an Air-pump or Condenser;* ρ, ρ_n *the densities of air in the receiver at first and after* n *descents of the piston: then in the*

Air-pump, $\rho_n = \left(\dfrac{R}{R+b}\right)^n \rho$; *Condenser,* $\rho_n = \dfrac{R+nb}{R}\rho$.

7. If the density of air in the receiver of an air-pump which has only one barrel, be diminished to one-fourth of its original density after 3 turns; compare the capacities of the receiver and barrel.

8. If in an air-pump, the density after 5 turns is to the original density as 7 : 44; compare the capacities of the receiver and barrel.

9. If there be two air-pumps with receivers, each of 10 cubic feet, and the single barrels be of 1 and 2 feet capacity respectively; compare the quantities of air exhausted by 5 turns of the first, and 3 turns of the second.

10. If a body, when placed under the receiver of a given air-pump, weigh a oz., and after n turns weigh a' oz., determine the weight

Ex. 8.

of the body in a vacuum; and if the Sp. Gr. of the body be given, find the density of the air in the receiver at first.

11. Having given the quantity of air p contained in the air-pump at first; how many turns will be required to exhaust a given quantity q?

12. If the mercury in a barometer, placed in the receiver of an air-pump, stand at 30 inches; and after 12 turns has sunk to 17 inches; compare the capacities of the receiver and barrel.

13. Having given the altitude k of the mercury in the barometer-gauge of an air-pump, and the capacities of the receiver and barrel; find the number of turns.

14. A barometer, having some air in the tube l, stands at an altitude a, and being placed under the receiver of an air-pump in which $R = mb$, after n turns the mercury has an altitude c; find the standard altitude and the quantity of air in the tube.

15. The barrel of a condenser is equal to each of two barrels of an air-pump; and R the receiver, which is equal to the receiver of the air-pump, $= 20b$; if each be worked 4 turns, compare the densities of the air then contained in the receivers.

16. If the barrel of an air-pump discharges at every turn into the receiver of a condenser; determine the density of the air in the condenser after n turns, both vessels being filled with common air at first.

17. A receiver whose capacity is R, has two barrels connected with it; one of which, whose capacity is a, condenses; the other, whose capacity is c, exhausts: they take their strokes alternately, beginning with a; find the density of air in the receiver after n strokes of both.

18. If the capacity of the receiver of a condenser be 30 times that of the barrel, and the length of a horizontal gauge be 20 inches; determine the position of the globule of mercury after 12 turns.

19. The capacities of the receiver and barrel of a condenser are 10 and 1 cubic feet respectively; in the receiver is placed a cylindrical tube 20 inches long, closed at its upper end, its lower and open end in contact with the surface of mercury in a vessel: determine the ascent of the mercury in the tube at each of the first 5 successive descents of the piston; the barometer standing at 30 in.

20. The gauge of a condenser, being a cylindrical tube, as in Ex. 19, is one foot long; the space occupied by the air in it after 2 descents of the piston is 6 inches: find the space which the air will occupy after the 3rd descent of the piston, the barometer standing at 30 in.

21. If R, b denote the capacities of the receiver and barrel respectively of an air-pump; it is required to find the depths at which

Ex. 8.

the piston-valve will open on the 1st, 2nd, 3rd, &c. descents of the piston; the range of which is a.

22. If h, h' be the altitudes of the mercury in a barometer, placed in a cylindrical diving-bell of length a, at the beginning and end of a descent; find the depth descended by the bottom of the bell, σ being the density of mercury.

23. If a hemispherical diving-bell be sunk in water, until the surface of the water in the bell bisects its vertical radius; find the distance between the surfaces of the water, within and without the bell*.

24. If a prismatic diving-bell of given volume V full of air, be sunk to a depth so that the distance between the surfaces of the water within and without the bell is k; find the volume of air (at its natural density) which will be required to be forced into the bell, in order that $\frac{1}{n}$th of its volume may be kept free from water.

25. In two common pumps, each consisting of one uniform cylinder, with a valve at or near the surface of the water in the reservoir, if the greatest altitude of the piston above that surface be 20 feet in each; and the least altitude of the piston be 16 feet in one, and 17 feet in the other: find in each case the greatest height to which water can be raised*.

26. If c be the greatest distance between the piston and the surface of the water in a common pump, a the altitude of a column of water which the air would support; show that the pump cannot work unless the length of the stroke be greater than $c^2 \div 4a$.

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27. Compare the lengths of a degree on Fahrenheit's, the Centigrade and Reaumur's thermometers.

28. If 68° Fahr. $= m°$ Cent. $= n°$ Reaum.; find m and n.

29. Find the degrees on Fahr. corresponding to 25° Reaumur.

30. If the tube of a thermometer be ·1 inch in diameter, and the distance between the boiling and freezing points be 7 inches, determine the capacity of the bulb or volume below the freezing-point; assuming the expansion of mercury to be ·0001 for 1° Fahr.

31. A thermometer, open at the top, is filled with mercury, which weighs 1250 grains, and the temperature of which is 32°; on being exposed to a higher temperature, 4·5 grains of mercury are expelled; find this temperature, the expansion of mercury in volume being ·018 from 32° to 212°.

32. What part of its volume at 60°, is the expansion of a body for each additional degree of temperature; if it expand ·005 parts of the volume, which it has at 32° for each degree above 32°?

* *It is assumed that the water-barometer stands at 33 feet.*

Ex. 8.

33. Two thermometers are differently graduated; one of them denotes two particular temperatures by a° and b°, and the other by c° and d°; what will the latter indicate, when the former indicates n°?

HYDRODYNAMICS.

Efflux of Fluids from Vessels.

Ex. 9.

1. A cylinder, whose diameter is 5 inches and altitude one foot, is filled with fluid issuing into it through an aperture ·125 inch diameter, in $1\frac{1}{2}$ minutes; find the velocity of the fluid at the aperture.

2. If a column of fluid *immediately* over a small orifice, and having the same sectional area, be to the column which issues in one second as 2 : 5; find the velocity at the orifice, and the height of the fluid above it.

3. A cylindrical vessel filled with fluid rests with its base on a horizontal plane; through an orifice 3 feet from the base, the fluid spouts to a distance of 5 feet; or 7, or 10 feet on the horizontal plane; find the altitude of the cylinder corresponding to the three cases.

4. If a cylinder, filled with water, be placed upon a wall 9 feet in height; and at 2 feet from its base the water spouts through a small orifice and falls on the ground at the distance of 14 feet from the wall; determine the altitude of the cylinder.

5. If in the vertical side of a prismatic vessel 12 feet long filled with fluid, there be made two holes at the depths of 5 and 7 feet; find where the effluent streams will intersect.

6. If a cylinder 18 feet long and 5 feet in diameter, full of water, be inclined to the horizon at an angle of 60°, and a small orifice be made in the middle of it; find the range of the issuing stream on the horizontal plane on which the cylinder rests.

7. The velocity of water, issuing through a small orifice into a vacuum, is 5 times greater than when the pressure of the air is removed from the upper surface of the fluid; determine the depth of the orifice.

8. If a paraboloid, whose generating curve has for its equation $y^4 = 16x$, resting with its base on a horizontal plane, be kept constantly filled with fluid; find at what point a very small orifice must be made that the latus rectum of the parabola described by the issuing fluid may be 8.

9. If a cone, whose vertical angle is 90° and altitude 7 feet, be filled with fluid, and rest with its base on a horizontal plane; find the distance from the vertex, of an orifice in the side, so that the issuing fluid may just strike the base of the cone.

10. A sphere full of water is placed on a horizontal plane; find

Ex. 9.

where a small orifice must be made in it so that the parabola of the spouting fluid may just touch the surface of the sphere.

11. A cylindrical vessel filled with fluid rests with its base on a horizontal plane; find the position of the orifice that the range on the plane may be the greatest possible.

12. If a cylinder 10 feet long, filled with water, stand vertically on the top of a plane inclined at 30° to the horizon; find where a small orifice must be made in the cylinder that the issuing fluid may strike the plane at the greatest distance.

13. If the vertical angle of the cone in Ex. 9 be 60°, find the position of the orifice, that the fluid may strike the horizontal plane at the greatest distance from the base.

The symbol k is used to denote the area of the small orifice or of the vena contracta.

14. A cylindrical vessel empties itself in a certain time through a small orifice in the base: compare the volume of the fluid discharged with the volume which would have been discharged in the same time had the vessel been kept constantly full.

15. The orifices in the equal bases of two upright prismatic vessels are in the ratio of 2 : 1, and the vessels are emptied in equal times; compare their altitudes.

16. Divide a cylinder 14 feet long, filled with fluid, into two such parts, that the times of emptying the fluid contained in each, through a small orifice in the base, may be equal.

17. An upright cylindrical vessel empties itself through a small orifice in the base; compare the pressures upon the concave surface at first, and when half the time of emptying has elapsed.

18. A cylinder $2l$ feet long, has its lower half filled with mercury, and the rest with water. Find the time in which it will empty itself through a small orifice in its horizontal base; the Sp. Gr. of mercury being σ.

19. A small aperture is made in the vertical side of a cylindrical vessel filled with fluid, the diameter of which is to that of the orifice as $12\frac{1}{2} : 1$; compare the latus rectum of the parabola first described by the spouting fluid, with the length of a pendulum vibrating 3 times while the surface of the fluid descends to the orifice.

20. If the diameter of a cylinder be 10 inches, and the diameter of an orifice in its base ·025 in.; also the height of the fluid in the cylinder be $8\frac{1}{10}$ feet; find the time of emptying, taking $g = 32{\cdot}2$ ft.

21. If a cylinder of given dimensions, with its axis vertical, be filled with fluid, and the surface of the fluid descend through an nth part of the axis in t seconds; determine the diameter of the orifice in the base.

22. A cylinder filled with water empties itself through an orifice,

Ex. 9.

of radius a, in the side, at the height b. After the times r and s it is observed to spout to the distances m and n respectively from the foot of the cylinder; find the dimensions of the cylinder.

23. Find the time in which the surface of fluid in a conical vessel filled with fluid will subside to half its original altitude, through a small orifice in the vertex; the axis being vertical.

24. Compare the times of emptying two equal cones through equal orifices in the vertex and the base.

25. Find the time of emptying the frustum of a cone, the radii of whose ends are 5 and 9 inches, and altitude 2 feet, through a small orifice in its smaller end.

26. Find the time of emptying a square pyramid whose base is a^2 and altitude h, through a small orifice in the vertex, the axis being vertical.

27. A sphere is emptied, through a small orifice at the lowest point, in less time than any spherical segment of the same volume.

28. Find the time in which a hemisphere filled with fluid will empty itself through a small orifice in its vertex, the axis of the hemisphere being vertical.

29. Find the time in which a hemisphere filled with fluid will empty itself through a small orifice in the base, its axis being vertical.

30. If the times in which two hemispheres are emptied, one by an orifice in the vertex, the other by an equal orifice in the base, be as 3 : 5; find the ratio of their radii.

31. The times of emptying a segment of a sphere through equal orifices in its vertex and base are as 2 : 3, the base being horizontal in both cases; compare the volume of the segment with that of the sphere.

32. Find the time of emptying a given paraboloid filled with fluid, through a small orifice in its vertex, the axis being vertical.

33. Compare the times of emptying two equal paraboloids, through equal orifices, one in the vertex, the other in the base.

34. Find the equation to the parabolic curve, which revolving about its axis would generate a vessel, such that the time of emptying it would be to the time of emptying the circumscribing cylinder as 1 : 9.

35. If a prolate and oblate spheroid have the same axes $2a$, $2b$; compare the times of emptying through equal orifices in the extremities of the axes of revolution.

36. Find the time of emptying an ellipsoid, filled with fluid, through a given small orifice at the extremity of one of its principal axes $2c$, which is vertical.

37. Find the time of emptying a vessel, formed by the revolution of a given cycloid about its axis, through an orifice in its vertex, the axis being vertical.

Ex. 9.

38. A clepsydra in the form of a vessel of revolution, is constructed so that the water may descend through equal depths in equal times; investigate the equation to the generating curve.

39. Find the velocity with which water issues through a small orifice, 25 feet below its surface, into a vessel containing air, of which the density is one-third that of the atmosphere; when the water-barometer stands at 33 feet.

The height of the homogeneous atmosphere is assumed to be 27690 feet.

40. Find the velocity with which air rushes through a small aperture into a vacuum.

41. A closed paraboloid, with its axis vertical and vertex downwards, containing air of the natural density, is let down in water to a certain depth, and a small orifice being opened at its vertex, the water rises up to the middle point of its axis; find the depth, and the velocity with which the water first rushed in.

42. An air-pump has its receiver and single barrel of 10 and $1\frac{1}{2}$ cubic feet capacity respectively; after 15 descents of the piston, the external air is admitted through a small aperture into the receiver; find the initial velocity.

43. An air-pump and a condenser have each a receiver and barrel of 10 and 1 cubic feet respectively; a tube furnished with a stop-cock connects the receivers; after 12 descents of both pistons the stop-cock is opened; determine the initial velocity of the air from the receiver of the condenser to that of the air-pump.

RESISTANCES.

Ex. 10.

1. A lamina, in the form of a semicircle, moves through a fluid in the direction of its axis, first with its vertex, and next with its diameter foremost, compare the resistances in the two cases.

2. A lamina, in the form of a segment of a circle, moves through a fluid in the direction of its axis, first with its vertex, and next with its base foremost; compare the resistances in the two cases.

3. A lamina, in the form of a semi-ellipse, bounded by the minor axis, moves through a fluid in the direction of its major axis, first with its vertex, and next with its base foremost; compare the resistances in the two cases.

4. A lamina, in the form of a parabola, bounded by an ordinate, moves through a fluid in the direction of its axis, first with the vertex, and next with its base foremost; compare the resistances in the two cases.

5. If the lamina be in the form of a complete cycloid, and move as in the preceding Exs.; compare the resistances.

Ex. 10.

6. Compare the resistance on a sphere, which moves through a fluid, with the resistance on a circular plate of the same radius which moves with the same velocity in a direction perpendicular to its plane.

7. A solid segment of a sphere is placed in a stream which moves in the direction of its axis, first with its vertex, and next with its base opposed to the stream; compare the resistances in the two cases.

8. Compare the resistance on a given cylinder moving through a fluid in the direction of its axis, with the resistance on the same cylinder moving with the same velocity in a direction perpendicular to its axis.

9. A cone moves through a fluid with a given velocity in the direction of its axis, first with its vertex, and next with its base foremost; compare the resistances in the two cases.

10. If a paraboloid of revolution, whose base is perpendicular to its axis, move through a fluid in the direction of the axis, first with the vertex, and next with the base foremost; compare the resistances in the two cases.

11. A prolate spheroid moves through a fluid with a given velocity in the direction of its axis of revolution; determine the resistance.

12. A solid, generated by the revolution of a cycloid about its axis, moves with a given velocity through a fluid in the direction of the axis, first with its vertex, and next with its base foremost; compare the resistances in the two cases.

ANSWERS.

ANSWERS TO EXAMPLES.

ARITHMETIC.

Ex. 1.

[1] $\frac{5}{12}$; $\frac{8}{27}$; $\frac{7}{90}$; $\frac{13}{23}$; $\frac{7}{11}$. [2] $\frac{7}{8}$; $\frac{13}{15}$; $\frac{3}{17}$; $\frac{2}{7}$; $\frac{283}{343}$.

[3] $\frac{10}{11}$; $\frac{9}{11}$; $\frac{355}{113}$; $\frac{99991}{97073}$.

Ex. 2.

[1] $\frac{23}{12}$; $\frac{500}{9}$; $\frac{14443}{8}$; $\frac{2519}{120}$. [2] $\frac{58}{15}$; $\frac{96}{17}$; $\frac{167}{19}$; $\frac{179}{18}$; $\frac{293}{21}$.

Ex. 3.

[1] $\frac{21}{16}$; $\frac{3}{2}$; $\frac{4}{1}$. [2] $\frac{8}{11}$; $\frac{2}{1}$; $\frac{3}{5}$. [3] $\frac{3}{1}$; $\frac{1}{3}$.

Ex. 4.

[1] $1\frac{1}{4}$; $2\frac{23}{24}$. [2] $2\frac{11}{36}$; $\frac{1}{7}$. [3] $10\frac{5}{18}$; $15\frac{47}{924}$. [4] 1; $53\frac{11}{20}$.

Ex. 5.

[1] $1\frac{13}{105}$; $2\frac{43}{60}$. [2] $\frac{5}{18}$; $\frac{9}{560}$. [3] $2\frac{4}{5}$; $\frac{1}{8}$. [4] $2\frac{41}{45}$; $24\frac{37}{72}$.

Ex. 6.

[1] $1\frac{13}{36}$; $5\frac{15}{19}$; 2821. [2] $4\frac{5}{7}$; $\frac{33}{140}$; $\frac{465}{121808}$. [3] $15\frac{2}{7}$; $9\frac{33}{98}$.

Ex. 7.

[1] 10; $\frac{6}{13}$; $\frac{10}{21}$; 7; $\frac{1}{5}$. [2] $11\frac{77}{108}$; 2; $1\frac{1}{14}$; $35\frac{7}{25}$.

[3] $8\frac{10}{11}$; $\frac{11}{135}$. [4] $38\frac{2}{11}$; $\frac{11}{112}$.

Ex. 8.

[1] 9; $2\frac{1}{5}$; $3\frac{2}{3}$. [2] $\frac{64}{85}$; $\frac{113}{123}$. [3] $1\frac{1}{8}$; $\frac{53}{122}$; $\frac{7}{11}$.

[4] $1\frac{83}{820}$; $\frac{2}{3}$; 1. [5] $2\frac{11}{14}$; $5\frac{94}{399}$; $6\frac{97}{132}$. [6] $\frac{5}{57}$; $\frac{49}{198}$.

[7] 1. [8] $\frac{4}{7}$; $\frac{37}{975}$. [9] 5. [10] $\frac{82}{151}$; $\frac{68}{157}$; $\frac{7}{10}$.

Ex. 9.

[1] 2*s*. 3*d*.; £2 17*s*. $7\frac{1}{5}$*d*.; 7*s*. $7\frac{3}{4}$*d*.

[2] 42 lb.; 7 oz. 4 dwt.; 13 dwt. $13\frac{35}{121}$ gr.

[3] 7 cwt. 3 qr. 14 lb.; 2 R. 20 P.; 12 h. 51 m. $25\frac{5}{7}$ sec.

Ex. 9.

[4] £2 6*s*. $4\frac{2}{3}$*d*.; £49 4*s*. $7\frac{8}{11}$*d*.; £2 16*s*. 3*d*.
[5] £283 16*s*. $7\frac{1}{2}$*d*.; £2007 6*s*. $5\frac{29}{44}$*d*.
[6] £45 18*s*. $11\frac{1}{4}$*d*.; £2314 17*s*. $10\frac{47}{160}$*d*.
[7] £2 10*s*. 9*d*.; £2 16*s*. 3*d*.
[8] 25 m. 5 f. 15 p. $4\frac{5}{14}$ yd.; 1 cwt.
[9] £4 15*s*. $3\frac{3}{4}$*d*.; 14*s*. $1\frac{23}{33}$*d*.
[10] £1 2*s*. [11] £1 6*s*. $11\frac{19}{22}$*d*.
[12] 10*s*. $11\frac{35}{72}$*d*. [13] £30 14*s*. $8\frac{2}{5}$*d*.
[14] £8 10*s*. $8\frac{17}{18}$*d*. [15] 7 yr. 24 wk. 1 d. 34 m.

Ex. 10.

[1] £2 10*s*. $6\frac{3}{4}$*d*.; £1 12*s*. $7\frac{1}{19}$*d*. nearly.
[2] 525; 2 wk. 1 d.
[3] 2 R. $19\frac{42}{67}$ P.; $2091\frac{7}{17}$ nearly.
[4] £16 8*s*. $1\frac{1}{2}$*d*.; £2 7*s*. $6\frac{7}{44}$*d*.
[5] 3 cwt. 1 qr. 14 lb.; 4 miles 2 f. 80 yd.

Ex. 11.

[1] $\frac{1}{3}$; $\frac{7}{18}$. [2] $\frac{379}{480}$; $\frac{4}{5}$. [3] $\frac{49}{15}$; $\frac{3}{2}$.
[4] $\frac{1}{330}$. [5] $\frac{11}{36}$. [6] $\frac{7}{9}$. [7] $\frac{38}{123}$.
[8] $\frac{4621}{17532}$. [9] $\frac{27}{32}$. [10] $\frac{20}{171}$; $\frac{64}{605}$.

Ex. 12.

[1] $\frac{1}{336}$; $\frac{3}{28}$. [2] $\frac{19}{50}$; $\frac{1}{4}$. [3] $\frac{1}{7}$; $\frac{4}{5}$. [4] $\frac{3}{4}$.
[5] $\frac{21}{64}$. [6] $\frac{143}{140}$. [7] $\frac{2}{15}$. [8] $\frac{19}{47}$. [9] $5\frac{155}{293}$.
[10] $2\frac{1001}{2720}$. [11] $4\frac{5275}{6048}$. [12] $\frac{7}{30}$. [13] $\frac{2639}{8748}$

Ex. 13.

[1] $\frac{20}{9}$. [2] $\frac{213}{440}$; $\frac{7}{90000}$. [3] 8*d*.

Decimal Fractions.

Ex. 14.

[1] 5032·08; 21·3978. [2] ·000376; ·0000037128; ·43204577.
[3] 30·7930896; 34·96818. [4] ·009287808; 1·045678375.

Ex. 15.

[1] 267; ·0104; 750. [2] 1240; 220; 30000.

[3] ·0032; ·03965. [4] 1989·2092; 237286.

[5] 32091·14782; 34·3168.

Ex. 16.

[1] ·08; ·152; 2·8; ·00125; ·0078125.

[2] ·625; ·008125; ·5136; ·0448; 13·34.

[3] 2·1875; ·98; $5{\cdot}\dot{3}$. [4] ·79375; ·5.

[5] ·1705; ·32. [6] ${\cdot}\dot{3}$; ${\cdot}\dot{1}4285\dot{7}$; ${\cdot}0\dot{6}$; ${\cdot}18\dot{7}$.

[7] ${\cdot}\dot{1}\dot{3}$; $2{\cdot}3\dot{4}\dot{5}$; $5{\cdot}\dot{0}4\dot{5}$; ${\cdot}12\dot{3}4\dot{5}$.

[8] ${\cdot}20\dot{4}3\dot{2}$; ${\cdot}\dot{5}2380\dot{9}$; ${\cdot}\dot{1}6129032258064\dot{5}$.

Ex. 17.

[1] $916{\cdot}5\dot{2}4806\dot{6}$. [2] $1{\cdot}8\dot{6}\dot{4}$; $17{\cdot}09\dot{5}822480747329\dot{9}$; ${\cdot}\dot{8}5714\dot{2}$.

[3] $35{\cdot}38\dot{3}7\dot{0}$; ${\cdot}\dot{3}9366\dot{6}$; $61{\cdot}\dot{1}6\dot{5}$. [4] ${\cdot}37\dot{5}9\dot{2}$; 54; $2{\cdot}\dot{6}$.

[5] ${\cdot}0069\dot{4}$; ${\cdot}01\dot{6}$; $3{\cdot}\dot{6}$. [6] 2·25: $1{\cdot}1\dot{4}\dot{5}$; $301{\cdot}\dot{7}1428\dot{5}$.

[7] ${\cdot}0\dot{5}\dot{2}$; 49; 5·6. [8] 61·99918 &c.; 1·683502 &c.

[9] 59·5875. [10] ·6065655.

Ex. 18.

[1] $\frac{1}{4}$; $\frac{29}{40}$; $\frac{817}{5000}$. [2] $1\frac{9}{25}$; $3\frac{291}{500}$; $15\frac{13}{50}$.

[3] $\frac{17}{200}$; $\frac{7}{800}$; $\frac{4707}{20000000}$. [4] $\frac{2}{9}$; $\frac{3}{11}$; $\frac{8}{33}$; $\frac{19}{11}$.

[5] $\frac{17}{330}$; $\frac{17}{54}$; $\frac{1}{88}$. [6] $\frac{4669}{900}$; $\frac{241}{27500}$; $\frac{745}{66}$. [7] $\frac{11}{9}$.

Ex. 19.

[1] 18·549; $18\frac{549}{1000}$.

[2] 7·525; 3·298; 43·05; $13{\cdot}\dot{6}$; 14601·3467075.

[3] 2·71805. [4] ·405465102.

[5] ·693146. [6] ·785398l4.

Ex. 20.

[1] ·375; ${\cdot}148958\dot{3}$. [2] ·91875; ·00625.

Ex. 20.

[3] $\cdot232\dot{5}\dot{7}$; $\cdot4097\dot{2}$. [4] $\cdot538\dot{5}8024691\dot{3}$; ·63.
[5] $\cdot19\dot{5}\dot{4}$. [6] 15·033707865 &c. [7] $\cdot282\dot{1}4285\dot{7}$.
[8] ·11025. [9] $\cdot0458\dot{3}$. [10] ·0390625.
[11] $\cdot\dot{7}$. [12] ·25. [13] ·042968.
[14] $\cdot053\dot{5}7142\dot{8}$. [15] ·2. [16] ·325.
[17] ·464197527 &c. [18] ·9004493 &c. [19] ·36752690 &c.

Ex. 21.

[1] 15*s.* 8·4*d.*; £3 9*s.* 3·6*d.*; 9·0522*d.*
[2] 17*s.* 6¾*d.*; 19*s.* 5·3184*d.*; £1 15*s.* 11·088*d.*
[3] 7*s.* 10½*d.*; 19*s.* 7·57*d.*; £2 1*s.* 8*d.*
[4] £4 14*s.* $8\frac{13}{16}$*d.*; £26 3*s.* 8¼*d.*; 15*s.* 7½*d.*
[5] 13·9968 gr.; 10·1376 dr.; 25 lb. 8 oz. 12·8 dr.
[6] 1 ft. 5·1 in.; 61 m. 377 yd. 2 ft. 6·96 in.; 79 yd. 2 ft. 5·376 in.
[7] 8 perches; 3 sq. ft. 99·4125 in.; 6 fur. 4 p. 4 yd. 1 ft. 2·4 in.
[8] £1 5*s.* $1\cdot58\dot{3}$*d.*; 7½*d.*; 7*s.* $0\frac{4}{5}$*d.*
[9] 1*s.* 3¾*d.*; £7 13*s.* $1\cdot38\dot{6}\dot{3}$*d.* [10] 17*s.* 0·6*d.*; £1 2*s.* 9¾*d.*
[11] 5*s.* 7·0104*d.* [12] 2*s.* 11¼*d.*
[13] £3 11*s.* 6·225*d.* [14] 3 cwt. 2 qr. 26 lb. ·8 oz.
[15] 2 ft. 3·609 in.; 3 furl. 25 poles; 3 furl. 10 p. 3 yd. 2 ft.
[16] 4 A. 1 R. 4 P. 29 yd. 8·864861 ft.

DUODECIMALS.

Ex. 22.

[1] Sq. 4 yd. 6 ft. 20 in. Sq. 48 yd. 4 ft. 132 in.
[2] Sq. 10 yd. 2 ft. 29 in. Sq. 180 yd. 7 ft. 54 in.
[3] Sq. 3 yd. 1 ft. 66⅔ in. Sq. 6 yd. 0 ft. 78 in.
[4] Sq. 11 yd. 8 ft. 54 in. 9′ 8″. [5] Sq. 361 yd. 3 ft. $86\frac{1}{12}$ in.
[6] Cub. 2 yd. 26 ft. 1680 in. [7] Cub. 50 yd. 25 ft. 423 in.

Ex. 23.

[1] 2 yd. 0 ft. 5⅓ in. [2] 4 yd. 2 ft. 8⅔ in.
[3] 12 yd. 1 ft. 5 in. [4] 21 yd. 0 ft. 8 in.
[5] 13 yd. 0 ft. 2½ in. [6] 69 yd. 0 ft. 9 in.

Practice.

Ex. 24.

[1] £1925 6*s*. $6\frac{1}{4}$*d*.; £93 19*s*. 11*d*.; £1906 8*s*. 9*d*.
[2] £529 11*s*. $0\frac{1}{2}$*d*.; £1603 0*s*. $5\frac{3}{4}$*d*.; £79 4*s*.
[3] £937 16*s*.; £17478 11*s*.; £41101 15*s*.
[4] £469 14*s*.; £4897 1*s*. 3*d*.; £696 1*s*. 11*d*.
[5] £16718 3*s*. $10\frac{1}{4}$*d*.; £993 15*s*.; £2540 6*s*.
[6] £698 5*s*. $7\frac{1}{2}$*d*.; £2770 9*s*. 11*d*.
[7] £1075 8*s*. 8*d*.; £5783 10*s*. $7\frac{1}{2}$*d*.
[8] £2749 4*s*. 11·352*d*.; £34620 14*s*. $6\frac{33}{64}$*d*.

Ex. 25.

[1] £10792 19*s*. $2\frac{1}{16}$*d*.
[2] £283 11*s*. $11\frac{5}{8}$*d*.
[3] £71 2*s*. 2*d*.
[4] £1 8*s*. $6{\cdot}41\dot{6}$*d*.
[5] £204 13*s*. $1\frac{1}{2}$*d*.
[6] £4 13*s*. $11\frac{1}{8}$*d*.
[7] £3 7*s*. $0\frac{3}{4}$*d*.
[8] £112 8*s*. $3\frac{1}{2}$*d*.
[9] £17 4*s*. $2\frac{2}{7}$*d*.
[10] £61 3*s*. $3\frac{27}{28}$*d*.
[11] £536 14*s*. 9*d*.
[12] £34 8*s*. 6*d*.
[13] £951 19*s*. 10*d*.
[14] £5118 2*s*. 5*d*.
[15] £136 15*s*. 4*d*.
[16] £21 19*s*. 1*d*.
[17] £880.

Ex. 26.

[1] £609 9*s*. $5\frac{129}{160}$*d*.
[2] £15 6*s*. 3*d*.
[3] £223 11*s*. 3*d*.

Rule of Three Direct.

Ex. 27.

[1] 2qr. 10oz. $4\frac{4}{7}$dr.
[2] £11 5*s*. 4·71*d*.
[3] $5\frac{11}{21}$ gall.
[4] £36 6*s*. $0\frac{1}{2}$*d*.
[5] £90.
[6] £494 8*s*.
[7] 11qr. 5bush. 2pk.
[8] 11.
[9] £15 19*s*. 6*d*.
[10] $6\frac{10860}{16493}$*d*.
[11] £200.
[12] £225 12*s*. 6*d*.
[13] 13*s*. 4·15*d*.
[14] 168 lb.
[15] $82\frac{2}{3}$.
[16] $10\frac{5}{7}$.
[17] $21\frac{3}{7}$.
[18] £16.
[19] 1*s*. $4\frac{1}{2}$*d*.
[20] $7\frac{1}{2}$ per cent. gain.
[21] £18 18*s*. $10\frac{3}{4}$*d*.
[22] £7 per cwt.
[23] £5 17*s*. $11\frac{5}{128}$*d*.
[24] $7\frac{1}{2}$; 3dwt. $8\frac{19}{40}$gr.
[25] ·4987 oz.
[26] 5 dwt. 3·2736 gr.

Rule of Three Inverse.

Ex. 28.

[1]	18 ft.	[2]	$63\frac{13}{18}$ yd.	[3]	$97\frac{7}{9}$ yd.
[4]	$6\frac{6}{7}$ mo.	[5]	7·1*d.*	[6]	$12\frac{12}{19}$*d.*
[7]	$214\frac{2}{7}$ miles.	[8]	$4166\frac{2}{3}$ yd.	[9]	$8\frac{19}{22}$ mo.
[10]	260 paces.	[11]	$42\frac{14}{33}$ paces.	[12]	1771.
[13]	$7\frac{1}{2}$ lb.				

Double Rule of Three.

Ex. 29.

[1]	£613 6*s.* 8*d.*	[2]	£453 12*s.*	[3]	£11 0*s.* $2\frac{2}{39}$*d.*
[4]	11 mo.	[5]	£6 0*s.* $3\frac{3}{4}$*d.*	[6]	$25\frac{109}{587}$ miles.
[7]	19·36 days.	[8]	88 A. 2 R. 15 P.	[9]	10 days.
[10]	9 days.	[11]	18 days.	[12]	7 hours.
[13]	6 days.	[14]	£15 16*s.* $0\frac{26}{27}$*d.*	[15]	£2000.
[16]	$23\frac{41}{408}$ loads.	[17]	200 of A.	[18]	358·6 days.

[19] 1. £1 13*s.* 6·82*d.*; 2. £1 10*s.* 1·29*d.*; 3. £1 9*s.* 2·52*d.*; 4. £1 9*s.* 2·18*d.*; 5. 7*s.* 4·31*d.*

[20] 1. 2*s.* 0·607*d.*; 2. 1*s.* 11·258*d.*; 3. 1*s.* 11·146*d.*; 4. 1*s.* 9·948*d.*

Simple Interest.

Ex. 30.

[1]	£35.	[2]	£291 19*s.* $7\frac{1}{2}$*d.*
[3]	£40 17*s.* 10·32*d.*	[4]	£35 7*s.* $1\frac{4}{5}$*d.*
[5]	£76 11*s.* 11·12*d.*	[6]	£20 10*s.*

Ex. 31.

[1]	£534 12*s.* 6*d.*	[2]	£1873 8*s.* $10\frac{4}{5}$*d.*
[3]	£744 16*s.* $1\frac{1}{8}$*d.*	[4]	£2249 13*s.* $2\frac{1}{2}$*d.*
[5]	£416 10*s.* $3\frac{1}{2}$*d.*	[6]	£3529 11*s.* $2\frac{9}{10}$*d.*

Ex. 32.

[1]	£10 8*s.* 6*d.*;	£357 18*s.* 6*d.*
[2]	£54 3*s.* 0·61*d.*;	£739 1*s.* 8·61*d.*
[3]	£1 12*s.* 9*d.*;	£128 2*s.* 9*d.*
[4]	£43 19*s.* $4\frac{1}{2}$*d.*;	£253 19*s.* $4\frac{1}{2}$*d.*
[5]	£112 7*s.* $5\frac{13}{73}$*d.*;	£7612 7*s.* $5\frac{13}{73}$*d.*
[6]	£1 17*s.* $7\frac{1}{2}$*d.*;	£227 10*s.* $1\frac{1}{2}$*d.*

Ex. 33.

[1] $3\frac{1}{2}$ years. [2] 17 yr. [3] $5\frac{2}{3}$ yr.
[4] 180 yr. [5] $28\frac{4}{7}$ yr. [6] $47\frac{1}{17}$ yr.

Ex. 34.

[1] $2\frac{8}{21}$ per cent. [2] $3\frac{1}{2}$. [3] $4\frac{5}{8}$.
[4] $3\frac{1}{3}$. [5] 6.

Ex. 35.

[1] £92 10*s.* [2] £483 17*s.* $5\frac{1}{31}$*d.*
[3] £91 13*s.* 4*d.* [4] £14000.
[5] £560.

Compound Interest.

Ex. 36.

[1] £55 5*s.* $6\frac{1}{4}$*d.* [2] £97 8*s.* 1·11*d.*
[3] £50 12*s.* 4*d.* [4] £129 3*s.* $6\frac{1}{2}$*d.*

Ex. 37.

[1] £483 13*s.* $9\frac{3}{4}$*d.* [2] £329 0*s.* 9*d.*
[3] £980 12*s.* $4\frac{1}{2}$*d.* [4] £2846 15*s.* 1*d.*
[5] £268 13*s.* [6] £473 19*s.* 5*d.*

Ex. 38.

[1] £1 11*s.* $6\frac{3}{4}$*d.* [2] £350 8*s.* 5*d.*
[3] 13*s.* 0·1632*d.* [4] 3*s.* 10·807875*d.*

Ex. 39.

[1] £375. [2] £500.

Discount.

Ex. 40.

[1] £166 13*s.* 4*d.* [2] £21 3*s.* $4\frac{1}{4}$*d.*
[3] £30 7*s.* 6*d.* [4] £17 8*s.* $2\frac{1}{2}$*d.*
[5] £50 11*s.* $10\frac{6}{7}$*d.* [6] £1 0*s.* $6\frac{1}{8}$*d.*
[7] £1 9*s.* $9\frac{3}{4}$*d.*

Ex. 41.

[1] £840. [2] £765.
[3] £666 13*s.* 4*d.* [4] £296 6*s.* $2\frac{3}{4}$*d.*
[5] £400. [6] £1953 2*s.* 6*d.*

Stocks.

Ex. 42.

[1] £531 18s. $3\frac{1}{2}$d.
[2] £5050.
[3] £446 16s. $2\frac{2}{47}$d.
[4] £1601 19s. $10\frac{3}{4}$d.
[5] £650.
[6] £2935 10s. $10\frac{3}{4}$d.

Ex. 43.

[1] £821 5s.
[2] £408 6s. $0\frac{1}{2}$d.
[3] £9375.
[4] £3885.
[5] £4064 14s. $7\frac{1}{2}$d.

Ex. 44.

[1] £67 15s. $11\frac{11}{59}$d.
[2] £35.
[3] £140.
[4] £151 13s. 4d.
[5] £179 5s. $4\frac{16}{41}$d.

Ex. 45.

[1] £3 8s. $11\frac{17}{29}$d.; £3 4s.
[2] $77\frac{7}{9}$; £1542 17s. $1\frac{5}{7}$d.
[3] £5212 15s. $3\frac{39}{47}$d.; £6 2s. $4\frac{4}{47}$d.
[4] £113 9s. $1\frac{3}{4}$d.
[5] £53 6s. 8d.
[6] £2 increase.

Fellowship.

Ex. 46.

[1] A, £10 5s. $10\frac{10}{17}$d.; B, £11 15s. $3\frac{9}{17}$d.;
C, £13 4s. $8\frac{8}{17}$d.; D, £14 14s. $1\frac{7}{17}$d.
[2] £136 11s. $11\frac{49}{97}$d.; £163 18s. $4\frac{20}{97}$d.; £229 9s. $8\frac{28}{97}$d.
[3] A, £666 13s. 4d.; B, £333 6s. 8d.;
C, £166 13s. 4d.; D, £83 6s. 8d.
[4] $41\frac{2}{3}$; $31\frac{11}{24}$; $26\frac{7}{8}$.

Ex. 47.

[1] A, £212 2s. $6\frac{210}{409}$d.
B, £194 8s. $11\frac{397}{409}$d.
C, £176 15s. $5\frac{175}{409}$d.
D, £139 13s. $0\frac{36}{409}$d.

[2] Capt., £251 10s. $6\frac{1914}{4421}$d.
Lieut., £98 16s. $0\frac{1248}{4421}$d.
Serg., £76 0s. $0\frac{960}{4421}$d.
Corp., £26 1s. $1\frac{3487}{4421}$d.
Private, £8 13s. $8\frac{2636}{4421}$d.

Extraction of Roots.

Ex. 48.

[1] 51; 85; 99; 217; 372.

[2] 625; 744; 2401; 6561.

[3] 1·414213; 12·247448; 32·449961; 54·772255; 87·509999.

[4] 8·6802; ·6576; 29·606; ·027.

[5] 6·0000005208; 34·5761; ·30006249.

[6] 2·3094; 1·$\dot{3}$; 2·$\dot{4}$9; 42·29158.

[7] $\frac{5}{6}$; $5\frac{1}{4}$; $5\frac{2}{3}$; $16\frac{7}{12}$.

[8] ·881917; ·69189; ·9400155; 3·952847; 2·9032.

[9] ·24253; ·31943; 8·7649; 20·4939.

Ex. 49.

[1] 46; 74; 98; 144. [2] 512; 6·49.

[3] ·9226; 1·03; ·115304. [4] ·192; ·03859.

[5] 1·29266; 1·816; ·82207; ·9859.

Ex. 50.

[1] 9; 36; 6·9. [2] 9·05538; 1·0583; ·99.

Ex. 51.

[1] 9; 8; 5·1961524. [2] ·25. [3] 20·5.

ALGEBRA.

Ex. 1.

[1] 46; $\frac{11}{9}$. [2] 7. [3] 0. [4] $\sqrt[3]{12}$.

Multiplication.

Ex. 2.

[1] a^3b^5c; $-6mnx^4y^6$; $h^2k^3t^3x^{10}$.

[2] $12x^5y^2-8x^4y^3+8x^3y^4-4x^2y^5$.

[3] $6x^2+13xy-5y^2$; $4x^3-9x^2y-9xy^2$.

[4] $36x^4+29x^2y^2-20y^4$.

[5] $a^4+a^2x^2+x^4$.

[6] $x^4-3x^3y-9x^2y^2+23xy^3-12y^4$.

[7] $x^5-(a+t)x^4+(b+at+s)x^3-(c+bt+as)x^2+(ct+bs)x-cs$.

[8] $27x^3-y^3+18xy+8$.

Ex. 2.

[9] $a^3+b^3+c^3-3abc.$

[10] $5x^4+\frac{7}{2}ax^3-\frac{107}{12}a^2x^2+\frac{5}{6}a^3x+\frac{7}{6}a^4.$

[11] $\frac{3}{4}h^5-4h^4+\frac{77}{8}h^3-\frac{43}{4}h^2-\frac{33}{4}h+27.$

[12] $x^6+(a^2-2b^2)x^4-(a^4-b^4)x^2-a^6-2a^4b^2-a^2b^4.$

[13] $2a^{2m}+2a^mb^p-4a^mc^n-3a^mb-3b^{p+1}+6bc^n.$

[14] $x^{mn}-x^{(m-1)n}y^m-x^ny^{(n-1)m}+y^{mn}.$

[15] $x^4+2x^3-41x^2-42x+360.$

[16] $4x^4-\frac{1}{4}.$ [17] $x^4-10x^3-37x^2+286x+840.$

[18] $9x^4-52x^2y^2+64y^4.$

DIVISION.

Ex. 3.

[1] $15a^3$; $21x^2y^5z^3.$ [2] $3y-5xy^3z+2x^3y^2z^2.$ [3] $2x-y.$

[4] $x^2+7x+9.$ [5] $3h^3-5h^2k+2hk^2.$

[6] $a^4+3a^3+9a^2+27a+81$; $x^{12}-x^9y^2+x^6y^4-x^3y^6+y^8.$

[7] $x^4+2ax^3+3a^2x^2+2a^3x+a^4.$ [8] $1+2x+3x^2+4x^3+5x^4.$

[9] $p+2q-r.$ [10] $x^2+2xy-5xz+4y^2+10yz+25z^2.$

[11] $x^3+12x^2-43x+30.$ [12] $x^2-2xy+\frac{1}{2}y^2.$

[13] $\frac{5}{6}a+\frac{3}{4}b+5c.$ [14] $bx^2+cx-f.$

[15] $(x^2+x+1)a-(x+1).$ [16] $2x^3y^{-3}-3x^4y.$

[17] $1+5x+15x^2+45x^3+135x^4+$ &c.

[18] $1+x+2x^2+8x^3+32x^4+128x^5+$ &c.

[19] $x^{pq-p}+x^{pq-2p}+x^{pq-3p}+$ &c. $+x^{2p}+x^p+1.$

[20] $(b-c)a^{n-2}+(b^2-c^2)a^{n-3}+(b^3-c^3)a^{n-4}+$ &c.
$+(b^{n-2}-c^{n-2})a+(b^{n-1}-c^{n-1}).$

GREATEST COMMON MEASURE.

Ex. 4.

[1] $2axy$; $3xz.$ [2] $5ax.$ [3] $2a-b.$ [4] $a+3b.$

[5] $x-1.$ [6] $3x+2.$ [7] $2x^2+7x+3.$

[8] $x^3-3x^2y+3xy^2-y^3.$ [9] $3x^2+a^2.$ [10] $3x-2.$ [11] $x^2-3.$

[12] $3a^2-2ab+b^2.$ [13] $2a^2x-3ay^2.$ [14] $2a+3b-c.$

[15] $x+a.$ [16] $ae^x+e^x+a+1.$ [17] $3x-y.$

Least Common Multiple.

Ex. 5.

[1] $56a^3b^2x^3$; $15x^2y^3-12x^3y^2$. [2] $72a^2x^5$; $160a^2x^6y^2(x-y)$.

[3] $6x^2-24y^2$; $a^4-a^3x+ax^3-x^4$.

[4] $21x^4-26x^3-55x^2+78x-24$. [5] $36ax^2(a^2-x^2)$.

[6] $x^5-3x^4-19x^3+3x^2+18x$.

[7] $12(x^5-2x^4y+x^3y^2+x^2y^3-2xy^4+y^5)$. [8] $x^{10}-x^6-x^4+1$.

Fractions.

Ex. 6.

[1] $\frac{15b^2-3ac^2}{5c}$; $\frac{7a}{5x}$; $\frac{11x}{x-a}$. [2] $\frac{x-1}{x+1}$; $\frac{3x+2}{2x^2+3x}$; $\frac{x^2-5x+5}{x^2-x+1}$.

[3] $\frac{x^2-7x+10}{x-10}$; $\frac{x^2-12x+35}{3x-17}$. [4] $\frac{5x^2+1}{9x^3-4x}$; $\frac{x^2+2xy+4y^2}{2(2x-3y)}$.

[5] $\frac{c+y}{f+2x}$; $\frac{c}{2df}$; $\frac{cx+d}{ax+b}$. [6] $\frac{4a(a+3b)}{7x^2(b+c)}$.

Ex. 8.

[1] $\frac{acx+b^2y}{bc}$; $\frac{x^2+y^2}{x^2-y^2}$; $\frac{a^2+b^2}{a^2-b^2}$. [2] $\frac{1}{x^2(x^2-1)}$; 0.

[3] $\frac{1}{1-x^4}$; $\frac{x^5+x^3+1}{x^2(x^2+1)^2}$. [4] $\frac{x}{4x^2-y^2}$; 1.

[5] $\frac{1+x+x^2}{1-x-x^4+x^5}$. [6] $\frac{1}{(x+2)^2(x^2+1)}$. [7] $\frac{a+bx}{c+dx}$; $\frac{a+bx}{b+ax}$.

[8] $\frac{-2x+2x^2-x^3}{(1-x)^4}$. [9] $\frac{x^2+x+1}{x^3(x^2+1)^2}$.

[10] $\frac{1}{(x+1)(x+3)}$. [11] $\frac{x}{(x+1)(x+2)(x+3)}$.

[12] $\frac{23+16x-30x^2-3x^3}{6-11x^3-21x^4-x^3+3x^4}$. [13] $\frac{3-3x-x^2+2x^3-x^4+x^5-x^6}{1+x+x^2+2x^3+x^4+x^5+x^6}$.

[14] $\frac{3-x^2-5x^3-4x^4-3x^5}{1+2x+3x^2+3x^3+2x^4+x^5}$. [15] $\frac{x+c}{(x-a)(x-b)}$.

[16] $\frac{1}{abc}$. [17] $\frac{1}{(x-a)(x-b)(x-c)}$. [18] $\frac{x}{(x-a)(x-b)(x-c)}$.

[19] $\frac{x^2}{(x-a)(x-b)(x-c)}$. [20] $\frac{x^2+hx+k}{(x-a)(x-b)(x-c)}$. [21] $x^{2n}+2$.

Ex. 9.

[1] $\frac{x^2}{y}$; $\frac{bcd}{a}$; $\frac{2a(cx-y)}{3c(x+y)}$. [2] $\frac{a^4-x^4}{a^2x}$; $\frac{a-x}{a+2x}$.

Ex. 9.

[3] $\dfrac{ax(a^2-ax-x^2)}{a^2-x^2}$; $\dfrac{a^4+a^2x^2+x^4}{a^4-x^4}$. [4] $\dfrac{x^2-11x+28}{x^2}$; $\dfrac{x+2}{x+3}$.

[5] $\dfrac{8ab}{9x^2}+2+\dfrac{9x^2}{8ab}$; $\dfrac{3a^4}{x^4}-\dfrac{19a^3b}{10x^3y}+\dfrac{21a^2b^2}{5x^2y^2}-\dfrac{9ab^3}{10xy^3}+\dfrac{b^4}{y^4}$.

[6] $\dfrac{a^4}{b^6}-\dfrac{4c^6d^8}{b^{10}}-\dfrac{14c^5d^4}{a^4b^8}-\dfrac{49c^4}{4a^8b^6}$. [7] $rs+(rt+qs)x+qtx^2$.

[8] $\dfrac{2y}{b}\left(\dfrac{x}{a}-\dfrac{z}{c}\right)$. [9] $2\left(\dfrac{bc}{ad}+\dfrac{ad}{bc}\right)$.

Ex. 10.

[1] $\dfrac{25a^2d}{3bc^2}$; $\dfrac{ac}{ab-bx}$; $\dfrac{a^2+b^2}{(a-b)^2}$. [2] $\dfrac{4x^3-3xy^2}{9x^3+8y^3}$; $\dfrac{fh(ad+bc)}{bd(eh-fg)}$.

[3] 1; $\dfrac{a+x}{x}(a^2+ax+x^2)$. [4] 1. [5] $\dfrac{1}{2x^2-1}$.

[6] $\dfrac{x^2+y^2}{2xy}$. [7] $\dfrac{twy(adf-bcf+bde)}{bdf(swy+twx+tyz)}$. [8] $\dfrac{a}{c}-\dfrac{b}{d}+\dfrac{c}{e}$.

[9] $\dfrac{acx^2}{bd}+\dfrac{bx}{cd}+\dfrac{a}{b}$. [10] $\dfrac{a-bx}{a+bx}$.

Ex. 11.

[1] a. [2] 1. [3] $\dfrac{x^3+5x^2-29x-105}{3x^2+10x-29}$; $\dfrac{x+3x^2+2x^3+x^4}{1+4x+3x^2+2x^3}$.

[4] $\dfrac{47-4x}{12x-5}$; $\dfrac{ab(1-x^2)}{(b^2-a^2)x}$.

Ex. 12.

[1] -3. [2] 3; $\dfrac{9}{85}$. [3] -1. [4] $2a$.

INVOLUTION AND EVOLUTION.

Ex. 13.

[1] $9a^2x^4z^6$; $-125a^6b^3x^9$; $\dfrac{256p^{12}q^{20}}{81r^8}$.

[2] $x^2-10xy+25y^2$; $9x^2+12xy+4y^2$; $x^4-6x^2y^2+9y^4$.

[3] $1-4x+10x^2-12x^3+9x^4$; $x^6-4x^5+6x^4-8x^3+9x^2-4x+4$.

[4] $4x^2+20xy-12xz+25y^2-30yz+9z^2$;

$9a^2x^2-12abxy+6acxz+4b^2y^2-4bcyz+c^2z^2$.

[5] $\dfrac{x^2}{y^2}-6+\dfrac{9y^2}{x^2}$; $\dfrac{x^2}{4}-\dfrac{xy}{3}+\dfrac{xz}{4}+\dfrac{y^2}{9}-\dfrac{yz}{6}+\dfrac{z^2}{16}$;

$\dfrac{a^4}{x^2}-\dfrac{4a^2b^2}{xy}+\dfrac{6a^2c^2}{xz}+\dfrac{4b^4}{y^2}-\dfrac{12b^2c^2}{yz}+\dfrac{9c^4}{z^2}$.

Ex. 13.

[6] $8x^6-36x^4y^2+54x^2y^4-27y^6$; $\frac{x^3}{a^3}+3\frac{x}{a}+3\frac{a}{x}+\frac{a^3}{x^3}$; $e^{3x}-3e^x+3e^{-x}-e^{-3x}$.

[7] $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$; $\frac{x^6}{y^6}-6\frac{x^4}{y^4}+15\frac{x^2}{y^2}-20+15\frac{y^2}{x^2}-6\frac{y^4}{x^4}+\frac{y^6}{x^6}$. [8] $(a+b+c)^3$.

[9] $16x^4-32x^3+24x^2-8x+1$; $x^5-15x^4y+90x^3y^2-270x^2y^3+405xy^4-243y^5$; $x^{12}-\frac{6ax^{10}}{c}+\frac{15a^2x^8}{c^2}-\frac{20a^3x^6}{c^3}+\frac{15a^4x^4}{c^4}-\frac{6a^5x^2}{c^5}+\frac{a^6}{c^6}$.

[10] $p^5x^5\mp 5p^4qx^4z+10p^3q^2x^3z^2\mp 10p^2q^3x^2z^3+5pq^4xz^4\mp q^5z^5$; $x^8-4x^6+10x^4-16x^2+19-\frac{16}{x^2}+\frac{10}{x^4}-\frac{4}{x^6}+\frac{1}{x^8}$.

Ex. 14.

[1] $\pm 3a^3x^2y^4$; $\pm\frac{8xy^5}{5z^3}$; $\pm\frac{12a^6b^2}{9c^3d^7}$. [2] $3a+b$; $4x-5y$.

[3] $2xz+3y$; $6a^2xy-\frac{1}{12ayz}$. [4] $1+x+3x^2$; $2x^2-3x+4$.

[5] $2a-3b+c$. [6] $4b^2-2ax+2x^2$. [7] $3x-y+5z+t$.

[8] $3x-5a-\frac{a^2}{2}$. [9] $x+\frac{a}{3}-\frac{b}{2}$. [10] $\frac{x}{5}+\frac{y}{6}-z$.

[11] $\frac{a}{x}+\frac{b}{2y}-\frac{3c}{z}$. [12] $\frac{x}{y}-\frac{1}{2}-\frac{y}{x}$. [13] $\frac{2x}{7y}-5+\frac{3y}{4x}$.

[14] $\frac{x^2}{2y^2}+1+\frac{2y}{x^2}$. [15] $x^3-2x^2y+3xy^2-4y^3$.

[16] $1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^8-$&c.; $x+\frac{a^2}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}-\frac{5a^8}{128x^7}+$&c.; $x-a-\frac{a^2}{2x}-\frac{a^3}{2x^2}-\frac{5a^4}{8x^3}-$&c.

Ex. 15.

[1] $3x^2y^5$; $\frac{7xz^2}{4y^4}$; $-\frac{2a^4x^6}{5y^3}$. [2] $1+2x$; $3x-2$.

[3] $2a^2-3x^2$. [4] $\frac{x^2}{y}-2y^2$; $\frac{x}{2}+\frac{2}{3x^2}$. [5] $\frac{3x}{y}-5-\frac{2y}{x}$.

[6] $x-4+\frac{2}{x}$. [7] $(a+1)^{2n}x-2ca^p$.

[8] $1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3-$&c.; $x^{\frac{2}{3}}-a^2x^{-\frac{4}{3}}-a^4x^{-\frac{10}{3}}-\frac{5}{3}a^6x^{-\frac{16}{3}}-$&c.

Ex. 16.

[1] $\frac{x}{y}-2y-\frac{y^3}{x}$. [2] $x^2-\frac{y}{2}$. [3] $a-\frac{b}{3}$.

SURDS.

Ex. 17.

[1] $24^{\frac{1}{3}}$; $250^{\frac{1}{2}}$; $\left(\frac{108}{13}\right)^{\frac{1}{2}}$; $\left(\frac{112}{45}\right)^{\frac{1}{3}}$.

[2] $(25a^4x)^{\frac{1}{2}}$; $-(16xy^2)^{\frac{1}{4}}$; $(-36a^2bc^2)^{\frac{1}{5}}$; $(x^2-a^2)^{\frac{1}{2}}$.

Ex. 18.

[1] $5\sqrt{5}$; $3\sqrt[3]{6}$; $2\sqrt[4]{3}$; $2\sqrt[5]{5}$; $36\sqrt{3}$.

[2] $4\sqrt[3]{4}$; $\frac{4}{27}\sqrt[4]{2}$; $\frac{3}{2}\sqrt[3]{75}$; $\frac{1}{4}\sqrt[4]{147}$.

[3] $6a^2xy^{\frac{3}{2}}$; $2ab(10a^2c^2)^{\frac{1}{3}}$; $2xy^2\left(\frac{x}{3y}\right)^{\frac{1}{4}}$.

[4] $\frac{y}{14z^2}(6xy)^{\frac{1}{2}}$; $\frac{1}{3}(30xz)^{\frac{1}{3}}$; $\frac{1}{3}(9ac)^{\frac{1}{4}}$.

Ex. 19.

[1] $\sqrt{2}$. [2] $\frac{1}{2}\sqrt{3}$. [3] $11\sqrt{3}$; $5^{\frac{4}{3}}$; $\sqrt[3]{9}$.

[4] $6ax^{\frac{1}{2}}$; $3a^2(3b)^{\frac{1}{2}}$. [5] $a^2b^{\frac{4}{3}}$; $\left(\frac{3}{2}a-\frac{2a^2}{9b}\right)b^{\frac{1}{3}}$.

Ex. 20.

[1] $2a^{\frac{13}{6}}$; $x^{\frac{1}{3}}y^{\frac{25}{12}}$; $\left(\frac{y}{x}\right)^{\frac{1}{6}}$. [2] $a^{\frac{3}{4}}x^{-\frac{1}{2}}-a^{\frac{25}{12}}x^{-\frac{5}{4}}+a^{\frac{13}{12}}x^{-\frac{1}{4}}-a^{\frac{1}{12}}x^{\frac{1}{4}}$.

[3] $x^2+6xz^{\frac{1}{3}}-4y+9z^{\frac{2}{3}}$. [4] $a-b$; $x^{\frac{1}{2}}y-y^{\frac{1}{3}}$.

[5] a^3-64b^2. [6] $x^2-3x^{\frac{4}{3}}y^{\frac{2}{3}}+3x^{\frac{2}{3}}y^{\frac{4}{3}}-y^2$.

[7] $x^{\frac{11}{12}}y^{\frac{13}{12}}+x^{\frac{13}{15}}y^{\frac{17}{15}}-x^{\frac{21}{20}}y^{\frac{19}{20}}-x^{\frac{17}{15}}y^{\frac{13}{15}}$.

[8] $\frac{1}{2}a^{\frac{3}{4}}-\frac{3}{10}a^{\frac{7}{10}}+\frac{1}{3}a^{\frac{7}{12}}-\frac{1}{5}a^{\frac{8}{15}}+\frac{1}{4}a^{\frac{1}{2}}-\frac{3}{20}a^{\frac{9}{20}}$.

[9] $a^{-\frac{1}{3}}-a^{\frac{1}{6}}b^{\frac{1}{3}}$. [10] $px^3-ax^2-a^2x-(p-2)a^3$.

[11] $a+a^{\frac{3}{4}}x^{-\frac{1}{4}}-a^{\frac{1}{4}}x^{-\frac{3}{4}}-x^{-1}$. [12] $(a^2-b^2)^{\frac{m+n}{mn}}$.

[13] 16. [14] $x^4+2x^3-8x^2-6x-1$.

[15] $-(ab)^{\frac{1}{2}}$; $\sqrt{-1}(ab)^{\frac{1}{4}}$; $\sqrt[3]{-1}(ab)^{\frac{1}{6}}$. [16] $16-(-3)^{\frac{1}{2}}$.

Ex. 21.

[1] 1; $\frac{1}{9}\left(\frac{1}{120}\right)^{\frac{1}{6}}$. [2] $\frac{3}{7}x^{\frac{13}{12}}$; $x^{\frac{3n-2m}{mn}}y^{n-m}$; $a^{-\frac{14}{9}}$. [3] $\frac{1}{18}a^{\frac{5}{4}}$.

[4] $8x^{\frac{3}{4}}+4x^{\frac{1}{2}}y^{\frac{1}{2}}+2x^{\frac{1}{4}}y+y^{\frac{3}{2}}$; $8x^{\frac{3}{4}}+2x^{\frac{1}{2}}y+\frac{1}{2}x^{\frac{1}{4}}y^2+\frac{1}{8}y^3$.

[5] $a^{\frac{5}{3}}+a^{\frac{4}{3}}b^{\frac{1}{6}}+ab^{\frac{1}{3}}+a^{\frac{2}{3}}b^{\frac{1}{2}}+a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{5}{6}}$; $a^{\frac{9}{4}}-a^{\frac{3}{2}}b^{\frac{3}{4}}+a^{\frac{3}{4}}b^{\frac{3}{2}}-b^{\frac{9}{4}}$.

[6] $x^{\frac{3}{2}}+2xy^{\frac{1}{2}}+4x^{\frac{1}{2}}y+8y^{\frac{3}{2}}$. [7] $a^2+ab^{\frac{2}{3}}+b^{\frac{4}{3}}$.

[8] $x^{\frac{5}{2}}-2x^{\frac{1}{2}}y^{\frac{3}{2}}-2y^{\frac{11}{6}}$. [9] $x-x^{\frac{1}{2}}$. [10] $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

[11] $x^{-\frac{2}{3}}+x^{-\frac{1}{3}}y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$;

$x^{-\frac{5}{2}}-2x^{-2}y^{\frac{1}{3}}+4x^{-\frac{3}{2}}y^{\frac{2}{3}}-8x^{-1}y+16x^{-\frac{1}{2}}y^{\frac{4}{3}}-32y^{\frac{5}{3}}$.

[12] $4x-2x^{\frac{1}{2}}y^{-\frac{1}{2}}+2x^{\frac{1}{2}}z^{\frac{1}{3}}+y^{-1}+y^{-\frac{1}{2}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$. [13] $p^{\frac{1}{3}}q^{\frac{1}{4}}-rp^{\frac{1}{5}}q^{\frac{1}{3}}$.

[14] $x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{2}x^{-\frac{5}{2}}+\frac{1}{4}x^{-\frac{7}{2}}+$ &c.

[15] $x^{\frac{1}{4}}+x^{-\frac{1}{4}}y^{\frac{1}{3}}+x^{-\frac{3}{4}}y^{\frac{2}{3}}+x^{-\frac{5}{4}}y+$ &c.

[16] $x^{\frac{n-1}{n}}+x^{\frac{n-2}{n}}a^{\frac{1}{n}}+x^{\frac{n-3}{n}}a^{\frac{2}{n}}+$ &c. $+x^{\frac{1}{n}}a^{\frac{n-2}{n}}+a^{\frac{n-1}{n}}$; n being integral.

[17] $x(3y)^{\frac{1}{2}}+y(2x)^{\frac{1}{2}}$.

Ex. 22.

[1] $a^{\frac{5}{12}}b^{\frac{73}{60}}$; $a^{\frac{1}{4}}+6ab^{\frac{1}{3}}c^{\frac{1}{12}}$. [2] $2\frac{(a^2-x^2)^{\frac{1}{2}}}{x^2}$; $(a^{-\frac{1}{n}}b^{\frac{1}{m}})^{pq}$.

[3] $\frac{1}{1-x^2}$. [4] $4x(x^2-1)^{\frac{1}{2}}$. [5] $2x^2$. [6] $\frac{x^3-2x^2+3x-4}{x^4-x^3+x^2-x+1}$.

[7] $\frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$; $\frac{ab^{\frac{1}{2}}}{b^{\frac{1}{2}}\mp c^{\frac{1}{2}}}$. [8] $\frac{3x^2}{x^3-1}$.

[9] $\frac{a^2+bc\sqrt{3}}{a-c}\left(\frac{ab}{3}\right)^{\frac{1}{2}}$. [10] $\left\{\left(\frac{y}{x}\right)^{\frac{1}{2}}+\left(\frac{x}{y}\right)^{\frac{1}{2}}+\frac{1}{2}\right\}^{\sqrt{2}}$.

Ex. 23.

[1] $-8x^{\frac{3}{2}}yz^{\frac{3}{4}}$; $16x^2y^{\frac{4}{3}}z$; $64x^3y^2z^{\frac{3}{2}}$.

[2] $-243x^{10}y^2z^{\frac{5}{3}}$; $-19683x^{18}y^{\frac{18}{5}}z^3$. [3] $\left(\frac{2x^2}{9y}\right)^{\frac{4}{3}}$; $a^{\frac{21}{5}}b^{\frac{7}{5}}c^{\frac{7}{30}}$.

[4] $ax^{-3}+3a^{\frac{1}{3}}x^{-1}+3a^{-\frac{1}{3}}x+a^{-1}x^3$;

$24x^3y(3y)^{\frac{1}{2}}-180x^{\frac{5}{2}}y^2+150x^2y^2(3y)^{\frac{1}{2}}-125x^{\frac{3}{2}}y^3$.

Ex. 23.

[5] $\frac{8}{27}x^{\frac{5}{2}}y^{-\frac{1}{2}}-2x^{\frac{3}{2}}y^{\frac{1}{2}}+\frac{9}{2}x^{\frac{1}{2}}y^{\frac{3}{2}}-\frac{27}{8}x^{-\frac{1}{2}}y^{\frac{5}{2}}$;

$a+b-c+3(a^2b)^{\frac{1}{3}}-3(a^2c)^{\frac{1}{3}}+(3ab^2)^{\frac{1}{3}}+3(ac^2)^{\frac{1}{3}}-3(b^2c)^{\frac{1}{3}}+3(bc^2)^{\frac{1}{3}}-6(abc)^{\frac{1}{3}}$.

[6] $2y\sqrt{-1}-3(x^2+y^2)^{\frac{1}{3}}\{(x+y\sqrt{-1})^{\frac{1}{3}}-(x-y\sqrt{-1})^{\frac{1}{3}}\}$.

[7] $a^3-4a^{\frac{9}{4}}b^{\frac{5}{2}}+6a^{\frac{3}{2}}b^5-4a^{\frac{3}{4}}b^{\frac{15}{2}}+b^{10}$;

$16x^6-96x^{\frac{9}{2}}y^{-\frac{3}{4}}+216x^3y^{-\frac{3}{2}}-216x^{\frac{3}{2}}y^{-\frac{9}{4}}+81y^{-3}$.

Ex. 24.

[1] $(135)^{\frac{1}{6}}$; $\frac{7^{\frac{1}{4}}}{3}\left(\frac{a^7}{b}\right)^{\frac{1}{6}}$. [2] $(3ax^{\frac{1}{3}})^{\frac{1}{2}}$; $(3ax^{\frac{1}{2}})^{\frac{1}{3}}$.

[3] $(-1)^{\frac{1}{m}}.2^{\frac{1}{m^2}}ab^2c^{\frac{2}{m}}$.

Ex. 25.

[1] $ax^{-1}+1+a^{-1}x$; $a^{\sqrt{2}}-a^{-\sqrt{2}}$.

[2] $5^{\frac{1}{2}}x^{\frac{3}{2}}-2c^{\frac{1}{2}}$; $1-\frac{3}{4}a^{\frac{1}{2}}+a$. [3] $\frac{3}{2}a^{\frac{3}{2}}-\frac{5}{3}ab^{\frac{1}{2}}+\frac{2}{5}a^{\frac{1}{2}}b$.

[4] $1-\frac{3}{4}x^{\frac{1}{3}}+x^{\frac{1}{2}}$. [5] $x-(xy-x^2)^{\frac{1}{2}}$; $a-(ax-a^2)^{\frac{1}{2}}$.

[6] $x+(a^2-x^2)^{\frac{1}{2}}$; $(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}$.

[7] $2x^{\frac{3}{2}}-y(x-y)^{\frac{1}{2}}$; $\left(\frac{1+m}{2}\right)^{\frac{1}{2}}+\left(\frac{1-m}{2}\right)^{\frac{1}{2}}$.

[8] $1-x+(1+2x-x^2)^{\frac{1}{2}}$; $(x+y)^{\frac{1}{2}}+z^{\frac{1}{2}}$.

[9] $x^{\frac{m}{2}}-\frac{1}{2}b^{\frac{1}{n}}x^{\frac{2}{p}}$; $a^{-\frac{1}{xy}}-a^{\frac{1}{2xy}}b^{\frac{1}{2x}}$.

Ex. 26.

[1] $\sqrt{5}+\sqrt{2}$; $2+\sqrt{3}$; $\frac{1}{\sqrt{2}}(\sqrt{3}+1)$; $3+\sqrt{7}$.

[2] $5-\sqrt{3}$; $\sqrt{5}-\sqrt{3}$; $6-\sqrt{5}$; $5-2\sqrt{3}$.

[3] $\frac{1}{\sqrt{2}}(\sqrt{7}-1)$; $2-\frac{1}{3}\sqrt{3}$; $\sqrt[4]{6}(1+\sqrt{2})$; $\sqrt[4]{3}(1+\sqrt{2})$.

[4] $\sqrt[4]{2}(\sqrt{2}-1)$; $\sqrt[4]{3}(\sqrt{5}-\sqrt{3})$; $\frac{1}{\sqrt[4]{2}}(\sqrt{2}+1)$; $\sqrt[4]{21}\left(1-\frac{1}{7}\sqrt{7}\right)$.

Ex. 27.

[1] $m+n+(m-n)\sqrt{-1}$; $n^{\frac{1}{2}}(1+\sqrt{-1})$.

Ex. 27.

[2] $2-\sqrt{-3}$; $1+\sqrt{-3}$; $2+\sqrt{-5}$; $\sqrt{-1}-\sqrt{-2}$.

[3] $5-2\sqrt{-1}$; $-1+2\sqrt{-1}$; $0{\cdot}2\pm 0{\cdot}1\sqrt{-1}$.

[4] $1+\sqrt{-1}$; $3(1-\sqrt{-1})$.

Ex. 28.

[1] $\sqrt{3}-\sqrt{2}-1$. [2] $1+\sqrt{3}+\sqrt{5}$.

[3] $1-\sqrt{2}+\sqrt{3}-\sqrt{6}$. [4] 6.

Ex. 29.

[1] $2a^{\frac{3}{2}}-3x^{\frac{3}{4}}$. [2] $xy^{-\frac{1}{3}}-x^{-\frac{1}{2}}y^{\frac{2}{3}}$. [3] $\frac{1}{2}x^{\frac{3}{4}}-5y^{\frac{4}{5}}$.

[4] $2+\sqrt{3}$; $\sqrt{2}+\sqrt{3}$; $\frac{1}{\sqrt[4]{2}}(\sqrt{3}+1)$; $\frac{1}{\sqrt[4]{3}}(\sqrt{2}-1)$.

[5] $2+\sqrt{-5}$; $a\sqrt{2}(1\pm\sqrt{-1})$; $2a^2(1\pm\sqrt{-1})$; $\frac{1}{\sqrt{2}}\pm\frac{1}{\sqrt{2}}\sqrt{-1}$.

Ex. 30.

[1] $\frac{1}{2}a-2b^{\frac{1}{2}}$. [2] $a^{-\frac{1}{2}}x^{\frac{1}{2}}-1+a^{\frac{1}{2}}x^{-\frac{1}{2}}$.

[3] $3(2a)^{\frac{1}{3}}-2(4x)^{\frac{1}{3}}$. [4] $x^{\frac{2}{3}}+a^{\frac{2}{3}}$.

[5] $2+\sqrt{5}$; $1+\sqrt{7}$; $1-\sqrt{2}$.

[6] $1-\sqrt{3}$; $\sqrt{21}-1$; $\sqrt{2}+\sqrt{3}$.

[7] $1+2\sqrt{-1}$; $2-\sqrt{-2}$; $\frac{1}{2}\pm\frac{\sqrt{3}}{2}\sqrt{-1}$.

[8] $1+\sqrt{2}$; $\sqrt{2}-1$.

Ex. 31.

[1] $\cdot 5$; $-\frac{9+2\sqrt{2}+3\sqrt{3}+2\sqrt{6}}{19}$; $\frac{2\sqrt{2}+\sqrt{3}}{5}$.

[2] $\sqrt{5}+\sqrt{2}$; $9+2\sqrt{15}$; $\frac{4\sqrt{5}-3\sqrt{2}}{2}$.

[3] $\frac{4+3\sqrt{2}-2\sqrt{3}-\sqrt{6}}{4}$; $\frac{3\sqrt{2}+2\sqrt{3}-\sqrt{30}}{12}$.

[4] $\frac{9\sqrt{3}-9\sqrt[3]{5}+3\sqrt[6]{16875}-15+5\sqrt[6]{675}-5\sqrt[3]{25}}{2}$; $\frac{5-8\sqrt[3]{3}+4\sqrt[3]{9}}{11}$;

$17-12\sqrt{2}+12\sqrt[3]{3}+8\sqrt[3]{9}-8\sqrt[6]{72}-6\sqrt[6]{648}$; $\frac{a^6+a^3b^{\frac{1}{3}}+b^{\frac{2}{3}}}{a^9-b}$.

Ex. 32.

[1] 1. [2] $a+b$. [3] b. [4] $\frac{1}{2}\left(xy+\frac{1}{xy}\right)$.

Ex. 33.

[1] $\left(\frac{2}{3}\right)^{\frac{2}{3}}$; $\sqrt[5]{2}$; $\sqrt[3]{9}$. [3] $\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}\sqrt{-1}$.

[4] $\{a-d+(b-c)\sqrt{-1}\}\times\{a-d-(b-c)\sqrt{-1}\}$.

I. Simple Equations.

Ex. 34.

[1] $x=2$. [2] $x=3$. [3] $x=6$. [4] $x=2$.

[5] $x=4$. [6] $x=1$. [7] $x=-4$. [8] $x=2$.

[9] $x=\frac{a+2c}{b-1}$. [10] $x=\frac{abc}{a+b}$. [11] $x=8$. [12] $x=42$.

[13] $x=12$. [14] $x=12$. [15] $x=120$. [16] $x=\frac{2}{3}$.

[17] $x=13$. [18] $x=5$. [19] $x=4$. [20] $x=9$.

[21] $x=\frac{bf(h-g)}{af+2bc-bfq}$. [22] $x=2a-5b+\frac{2b^2}{a}$. [23] $x=25a+24b$.

[24] $x=\frac{1}{ab}$. [25] $x=\frac{c}{2b}(3a-b)$. [26] $x=1\frac{13}{67}$. [27] $x=-8$.

[28] $x=72$. [29] $x=19$. [30] $x=8$. [31] $x=5\cdot1$.

[32] $x=2$. [33] $x=11$. [34] $x=\frac{1}{2}$. [35] $x=\frac{3}{5}$.

[36] $x=8$. [37] $x=1$. [38] $x=-107$. [39] $x=28$.

[40] $x=1\frac{5}{68}$. [41] $x=4$. [42] $x=4\frac{1}{3}$. [43] $x=\frac{7}{8}$.

Ex. 35.

[1] $x=4$. [2] $x=\frac{9}{20}$. [3] $x=\left(\frac{83}{30}\right)^2$. [4] $x=\frac{14}{17}$.

[5] $x=3$. [6] $x=\frac{1}{a}\left(b+\frac{c^2}{c-1}\right)^2$. [7] $x=\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}+2}$.

[8] $x=\frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}$. [9] $x=\frac{b}{2}\left(\frac{b-2a}{b-a}\right)$. [10] $x=\frac{16}{25}$. [11] $x=\frac{9a}{16}$.

[12] $x=\frac{64a}{1025}$. [13] $x=\frac{(n-1)^2}{2n-1}a$. [14] $x=\frac{961}{1025}a$, or $2a$.

[15] $x=\frac{24}{25}$. [16] $x=\frac{1}{4}\left(1-a+\frac{1}{1-a}\right)^2$. [17] $x=9$.

Ex. 35.

[18] $x=1$. [19] $x=\frac{a^2-b^2}{6a}$. [20] $x=a$. [21] $x=\pm 2a^{\frac{1}{2}}+a^{-1}$.

[22] $x=a\left\{1-\left(\frac{2b^{\frac{1}{2}}}{1+b}\right)^4\right\}$. [23] $x=\frac{4}{81}$. [24] $x=\frac{1}{2}(a^{\frac{1}{6}}-a^{-\frac{1}{6}})^2$.

II. Simultaneous Equations of the First Degree.

Ex. 36.

[1] $x=1$, $y=-1$. [2] $x=7$, $y=2$. [3] $x=4$, $y=7$.

[4] $x=3$, $y=7$. [5] $x=\frac{bc}{a+b}$, $y=\frac{ac}{a+b}$.

[6] $x=\frac{ac+b^2}{a^2+b}$, $y=\frac{ab-c}{a^2+b}$. [7] $x=\frac{11bc}{2a^2+3ab}$, $y=\frac{c}{b}\left(\frac{15b-a}{2a+3b}\right)$.

[8] $x=\frac{1}{2b}\left(\frac{n}{a-b}-c\right)$, $y=\frac{1}{2b}\left(c-\frac{n}{a+b}\right)$.

[9] $x=18$, $y=10$. [10] $x=144$, $y=216$. [11] $x=5$, $y=2$.

[12] $x=\frac{1}{4}$, $y=\frac{1}{5}$. [13] $x=99$, $y=15$. [14] $x=5$, $y=5$.

[15] $x=7$, $y=9$. [16] $x=\cdot02$, $y=2\cdot9$. [17] $x=2\frac{62}{65}$, $y=1\frac{29}{35}$.

[18] $x=\frac{m^2-n^2}{ma-nb}$, $y=\frac{m^2-n^2}{mb-na}$.

[19] $x=\frac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$, $y=\frac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}$.

[20] $x=21$, $y=20$. [21] $x=18$, $y=24$. [22] $x=3$, $y=2$.

[23] $x=\frac{ab}{a+b}$, $y=\frac{ab}{a-b}$. [24] $x=16$, $y=25$.

Ex. 37.

[1] $x=7$, $y=5$, $z=3$. [2] $x=1$, $y=2$, $z=4$.
[3] $x=5$, $y=6$, $z=7$. [4] $x=7$, $y=10$, $z=9$.
[5] $x=2$, $y=4$, $z=6$. [6] $x=2$, $y=-3$, $z=4$.
[7] $x=12$, $y=12$, $z=12$. [8] $x=5$, $y=7$, $z=-3$.
[9] $x=18$, $y=12$, $z=4$.

[10] $x=\frac{1}{(a-c)(b-c)}$, $y=\frac{1}{(a-b)(b-c)}$, $z=\frac{1}{(a-c)(a-b)}$.

[11] $x=\frac{2}{a+b-c}$, $y=\frac{2}{a-b+c}$, $z=\frac{2}{b+c-a}$.

[12] $x=\frac{12}{7}$, $y=\frac{12}{5}$, $z=-12$. [13] $x=-6\frac{6}{23}$, $y=2\frac{2}{71}$, $z=3\frac{21}{41}$.

Ex. 37.

[14] $x=\frac{1}{2}$, $y=\frac{1}{3}$, $z=\frac{1}{4}$. [15] $x=6$, $y=12$, $z=8$.

[16] $x=1$, $y=4$, $z=27$. [17] $x=5$, $y=4$, $z=3$, $u=2$, $t=1$.

III. Problems in Equations of the First Degree.

Ex. 38.

[1] 9. [2] 14. [3] 12. [4] 12. [5] 9, 5. [6] £8 15*s.*

[7] 88 yrs. [8] 8 hr.; 38 miles. [9] A, £70; B, £130. [10] 1000.

[11] $23\frac{1}{3}$ days. [12] $45\frac{1}{2}$, $10\frac{16}{53}$ days. [13] $2\frac{20}{53}$ days.

[14] $17\frac{5}{36}$, $14\frac{17}{36}$, $12\frac{5}{36}$ gall. [15] £1000, £1500, £2250, £3250.

[16] $\frac{4}{15}$. [17] 5,6. [18] 65. [19] Man, $21\frac{3}{7}$ days; woman, 50 days.

[20] A's : B's : C's $=m+1:n+1:p+1$ days. [21] £25.

[22] 21,40. [23] 1080 yd.; $16\frac{1}{2}$ m. [24] A horse, £24; a cow, £12.

[25] Barley, 28 bush.; Rye, 20 b.; wheat, 52 b.

IV. Pure Quadratic Equations.

Ex. 39.

[1] $x=\pm 3$. [2] $x=\pm\left(\frac{24}{89}\right)^{\frac{1}{2}}$. [3] $x=\pm 2\frac{1}{3}$.

[4] $x=\pm\left(\frac{55}{62}\right)^{\frac{1}{2}}$. [5] $x=\pm\cdot 3$. [6] $x=\pm 3$.

[7] $x=\pm 8$. [8] $x=\pm 5\cdot 5$. [9] $x=\pm 2$.

[10] $x=\pm\sqrt{5}$. [11] $x=\pm b\left(1-\frac{2a}{bc}\right)^{\frac{1}{2}}$. [12] $x=\pm\frac{1}{2}$.

[13] $x=\pm\frac{a}{2}\left(\frac{a^2-4}{a^2-1}\right)^{\frac{1}{2}}$. [14] $x=\pm\left(\frac{2ac}{bc^2+b}\right)^{\frac{1}{2}}$.

[15] $x=\pm(2ab-b^2)^{\frac{1}{2}}$. [16] $x=\pm\frac{2}{b(4a-b^2)^{\frac{1}{2}}}$.

[17] $x=\pm\left(\frac{a-2}{a+4}\right)^{\frac{1}{2}}$. [18] $x=\pm\frac{1}{a}\left(\frac{2a}{b}-1\right)^{\frac{1}{2}}$.

[19] $x=\pm\frac{\sqrt{3}}{2}a$. [20] $x=\pm n\left(a-\frac{n^2}{4}\right)^{\frac{1}{2}}$.

[21] $x=\pm\frac{4}{a^2}(a^2-4)^{\frac{1}{2}}$. [22] $x=\pm 2\left\{(1-a)(1-\frac{1}{3}a)\right\}^{\frac{1}{2}}$.

V. ADFECTED QUADRATIC EQUATIONS.

Ex. 40.

[1] $x=12, -2$. [2] $x=8, -10$. [3] $x=16, 2$. [4] $x=17, -4$.

[5] $x=107, -106$. [6] $x=4, -13$. [7] $x=136, -25$.

[8] $x=80, -75$. [9] $x=\frac{3}{10}, \frac{3}{2}$. [10] $x=3, -\frac{4}{3}$.

[11] $x=6, -4\frac{1}{2}$. [12] $x=5, -4$. [13] $x=\frac{1}{2}, -\frac{4}{3}$.

[14] $x=12\frac{1}{12}, -12$. [15] $x=2, 4$. [16] $x=\frac{3}{2}, -\frac{15}{22}$.

[17] $x=9\frac{15}{17}, -11$. [18] $x=7, -\frac{1}{3}$. [19] $x=2, \frac{1}{2}$.

[20] $x=\frac{1}{2}(5\pm\sqrt{22\cdot6})$. [21] $x=2, -\frac{2}{15}$. [22] $x=9, -\frac{55}{51}$.

[23] $x=3, -8\cdot7$. [24] $x=8\pm\sqrt{601}$. [25] $x=8, -\frac{4}{27}$.

[26] $x=3, -3\frac{23}{33}$. [27] $x=a, b$. [28] $x=1, \frac{2b}{a-b}$.

[29] $x=a, -b$. [30] $x=\frac{n}{q}, -\frac{p}{m}$. [31] $x=\frac{3}{4}a, \frac{1}{2}a$.

[32] $x=8, 4$. [33] $x=\frac{a}{2}(-3\pm\sqrt{3})$. [34] $x=\pm3, \pm\sqrt{-6\cdot8}$.

[35] $x=2, -1, -1\pm\sqrt{-3}, \frac{1}{2}(1\pm\sqrt{-3})$. [36] $x=2, (-6)^{\frac{1}{3}}$.

[37] $x=\frac{1}{4}, -\frac{1}{2}$. [38] $x=\frac{1}{a^2}, \frac{1}{4a^2}$. [39] $x=\left(\frac{1}{3}\right)^{\frac{2}{3}}, (-1)^{\frac{2}{3}}$.

[40] $x=-8, -\frac{1}{8}$. [41] $x=4, (-7)^{\frac{2}{3}}$. [42] $x=\frac{ac}{b}, \frac{bc}{a}$.

[43] $x=4, -1$. [44] $x=\pm5, \pm3\sqrt{2}$. [45] $x=1, 1\pm2\sqrt{15}$.

[46] $x=3, -\frac{1}{2}, \frac{1}{4}(5\pm\sqrt{1329})$. [47] $x=\frac{1}{2}(3\pm\sqrt{5}), \frac{1}{6}(9\pm\sqrt{-83})$.

[48] $x=a, 0, \frac{a}{2}\left\{1\pm\left(5-\frac{8b}{a}\right)^{\frac{1}{2}}\right\}$.

[49] $x=5, -6, \frac{1}{2}(\pm\sqrt{377}-1)$. [50] $x=\pm5, \pm4\sqrt{2}$.

[51] $x=6, -1, \frac{1}{2}(5\pm3\sqrt{-3})$. [52] $x=\frac{5}{2}, \frac{5}{8}$.

Ex. 40.

[53] $x=2\alpha\{\alpha\pm(\alpha^2+a)^{\frac{1}{2}}\}$, where $\alpha=\frac{1}{4}\{-a\pm(a^2+4b)^{\frac{1}{2}}\}$.

[54] $x=0,\ \frac{4c(1-c^2)}{(1+c^2)^2}$. [55] $x=6,\ -\frac{2}{3}$.

[56] $x=4,\ 9,\ \frac{1}{2}(-13\pm3\sqrt{-3})$. [57] $x=4,\ -2,\ -1\pm\sqrt{7}$.

[58] $x=\frac{1}{2}(3\pm\sqrt{29}),\ \frac{1}{2}(-3\pm\sqrt{21})$.

[59] $x=\frac{1}{2}(7\pm\sqrt{13}),\ \frac{1}{2}(-1\pm\sqrt{-3})$.

[60] $x=1,\ 16,\ \frac{1}{2}(1\pm3\sqrt{-7})$. [61] $x=3,\ -\frac{3}{4},\ \frac{9}{8}(9\pm\sqrt{97})$.

[62] $x=\pm\left(\frac{3\pm\sqrt{41}}{2}\right)^{\frac{1}{2}},\ \pm\left(\frac{1+\sqrt{37}}{2}\right)^{\frac{1}{2}}$. [63] $x=\frac{63}{65}a,\ 0$.

[64] $x=\pm a(\pm8\sqrt{2}-11)^{\frac{1}{2}}$. [65] $x=\pm\frac{1}{3}\sqrt{-3}$.

[66] $x=\pm\left\{\frac{2}{a}-\frac{1}{2}\pm\left(\frac{4}{a^2}-\frac{3}{4}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}$. [67] $x=\pm\left\{\frac{c+4\pm2\sqrt{3}(c+1)^{\frac{1}{2}}}{c-2}\right\}^{\frac{1}{2}}$.

[68] $x=\pm\left[a^2-\left\{h^2\pm\left(a+\frac{h^4}{2}\right)^{\frac{1}{2}}\right\}^4\right]^{\frac{1}{4}}$.

[69] $x=\pm\frac{1}{a}\left[-1\pm(1-a^4)^{\frac{1}{2}}\pm\{2\pm2(1-a^4)^{\frac{1}{2}}\}^{\frac{1}{2}}\right]^{\frac{1}{2}}$.

[70] $x=a,\ \frac{a}{8}(-3\pm\sqrt{-7})$. [71] $x=\left(\frac{a+b}{a-b}\right)^{\pm\frac{2pq}{p-q}}$.

[72] $x=\frac{3a}{32}(3\pm\sqrt{21})$.

Ex. 41.

[1] $x=8,\ 29\mp7\sqrt{-10}$. [2] $x=2,\ -1,\ -1$.

[3] $x=1,\ \frac{1}{4}(-1\pm\sqrt{-7})$. [4] $x=-3,\ \frac{1}{2}(3\pm\sqrt{-3})$.

[5] $x=4,\ 1\pm\sqrt{-1}$. [6] $x=1,\ \frac{1}{3}(6\pm\sqrt{-3})$.

[7] $x=4,\ 6\frac{1}{4},\ \frac{1}{8}(209\mp13\sqrt{249})$. [8] $x=4,\ -3,\ \frac{1}{2}(1\pm\sqrt{-43})$.

[9] $x=1,\ 1,\ \frac{1}{2}(-3\pm\sqrt{5})$.

Ex. 41.

[10] $x=\frac{1}{4}(3\pm\sqrt{-7}),\ \frac{1}{4}(-1\pm\sqrt{-15})$. [11] $x=5,\ -1,\ 2+\sqrt{5}$.

[12] $x=\pm 3,\ \frac{1}{6}(-13\pm\sqrt{-155})$.

[13] $x=-1,\ \dfrac{1+2h\pm\sqrt{3}(4h-1)^{\frac{1}{2}}}{2(1-h)}$. [14] $x=1\pm\sqrt{3}\pm(3\pm 2\sqrt{3})^{\frac{1}{2}}$.

[15] $x=\dfrac{1+4a+\sqrt{5}(1+4a)^{\frac{1}{2}}\pm\{-10+60a\pm 2\sqrt{5}(1+4a)^{\frac{3}{2}}\}^{\frac{1}{2}}}{4(1-a)}$.

VI. Simultaneous Equations of the 2nd, 3rd, &c. Degrees.

Ex. 42.

[1] $\left.\begin{array}{l}x=6,\ -4,\\ y=2,\ -3.\end{array}\right\}$ [2] $\left.\begin{array}{l}x=\pm 7,\ \pm 6,\\ y=\pm 6,\ \pm 7.\end{array}\right\}$ [3] $\left.\begin{array}{l}x=5,\ -4,\\ y=4,\ -5.\end{array}\right\}$

[4] $\left.\begin{array}{l}x=3,\ -2\frac{3}{17},\\ y=1,\ 2\frac{5}{17}.\end{array}\right\}$ [5] $\left.\begin{array}{l}x=7,\ -5,\\ y=2,\ -2.\end{array}\right\}$ [6] $\left.\begin{array}{l}x=9,\ -8\frac{1}{2},\\ y=-8,\ 9\frac{1}{2}.\end{array}\right\}$

[7] $\left.\begin{array}{l}x=5,\ \frac{3}{4},\\ y=3,\ -\frac{5}{4}.\end{array}\right\}$ [8] $\left.\begin{array}{l}x=\pm 5,\\ y=\pm 2.\end{array}\right\}$ [9] $\left.\begin{array}{l}x=3,\ -1\frac{26}{27},\\ y=-4,\ 8\frac{11}{27}.\end{array}\right\}$

[10] $\left.\begin{array}{l}x=3\frac{1}{2},\ -3\frac{32}{67},\\ y=2\frac{1}{4},\ -3\frac{153}{268}.\end{array}\right\}$ [11] $\left.\begin{array}{l}x=8,\ -8{\cdot}2,\\ y=12,\ -12{\cdot}3.\end{array}\right\}$ [12] $\left.\begin{array}{l}x=7,\\ y=\pm 4.\end{array}\right\}$

[13] $\left.\begin{array}{l}x=5,\ 1{\cdot}7,\\ y=3,\ -{\cdot}3.\end{array}\right\}$ [14] $\left.\begin{array}{l}x=7,\ -1,\ \pm\sqrt{-3}+3,\\ y=1,\ -7,\ \pm\sqrt{-3}-3.\end{array}\right\}$

[15] $\left.\begin{array}{l}x=4,\ 2,\ \frac{2}{3}(-2\pm\sqrt{-14}),\\ y=2,\ 4,\ \frac{2}{3}(-2\mp\sqrt{-14}.)\end{array}\right\}$ [16] $\left.\begin{array}{l}x=21\frac{3}{28},\\ y=6\frac{27}{28}.\end{array}\right\}$

[17] $\left.\begin{array}{l}x=3,\ 2,\ 5,\ 1,\\ y=2,\ 3,\ 1,\ 5.\end{array}\right\}$ [18] $\left.\begin{array}{l}x=7,\ 3,\ \frac{1}{6}(-31\pm\sqrt{205}),\\ y=3,\ 7,\ \frac{1}{6}(-31\mp\sqrt{205}).\end{array}\right\}$

[19] $\left.\begin{array}{l}x=\pm 2,\ \pm\frac{4}{3}\sqrt{3},\\ y=\pm 6,\ \pm\frac{10}{3}\sqrt{3}.\end{array}\right\}$ [20] $\left.\begin{array}{l}x=\pm\frac{1}{2}\{3a^2+b^2\mp(a^4+6a^2b^2+b^4)^{\frac{1}{2}}\}^{\frac{1}{2}},\\ y=\pm\frac{1}{2}\{a^2+3b^2\pm(a^4+6a^2b^2+b^4)^{\frac{1}{2}}\}^{\frac{1}{2}}.\end{array}\right\}$

[21] $\left.\begin{array}{l}x=\frac{ab}{b^2-a^2}\{b\mp(2a^2-b^2)^{\frac{1}{2}}\},\\ y=\frac{ab}{b^2-a^2}\{b\pm(2a^2-b^2)^{\frac{1}{2}}\}.\end{array}\right\}$ [22] $\left.\begin{array}{l}x=9,\ 3,\ -7\pm 2\sqrt{-1},\\ y=3,\ 9,\ -7\mp 2\sqrt{-1}.\end{array}\right\}$

Ex. 42.

[23] $\left.\begin{array}{l} x=16,\ 4, \\ y=4,\ 16. \end{array}\right\}$
[24] $\left.\begin{array}{l} x=16,\ 735\frac{1}{81}, \\ y=4,\ 723\frac{1}{81}. \end{array}\right\}$
[25] $\left.\begin{array}{l} x=4,\ -1, \\ y=1,\ -4. \end{array}\right\}$

[26] $\left.\begin{array}{l} x=27,\ 8, \\ y=8,\ 27. \end{array}\right\}$
[27] $\left.\begin{array}{l} x=9,\ 1, \\ y=1,\ 9. \end{array}\right\}$
[28] $\left.\begin{array}{l} x=5,\ 3, \\ y=3,\ 5. \end{array}\right\}$

[29] $\left.\begin{array}{l} x=\frac{1}{4b}\{a^2+b^2+(10a^2b^2-3a^4-3b^4)^{\frac{1}{2}}\}, \\ y=\frac{1}{4b}\{a^2+b^2-(10a^2b^2-3a^4-3b^4)^{\frac{1}{2}}\}. \end{array}\right\}$
[30] $\left.\begin{array}{l} x=5,\ 18\frac{1}{8}, \\ y=4,\ 17\frac{7}{8}. \end{array}\right\}$

[31] $\left.\begin{array}{l} x=4,\ 1,\ 0, \\ y=8,\ 0. \end{array}\right\}$
[32] $\left.\begin{array}{l} x=0,\ 4, \\ y=0,\ 3. \end{array}\right\}$
[33] $\left.\begin{array}{l} x=7,\ 2, \\ y=2,\ 7. \end{array}\right\}$

[34] $\left.\begin{array}{l} x=625,\ 1, \\ y=1,\ 3125. \end{array}\right\}$
[35] $\left.\begin{array}{l} x=4,\ 1,\ \frac{1}{2}(5\pm\sqrt{-159}), \\ y=1,\ 4,\ \frac{1}{2}(5\mp\sqrt{-159}). \end{array}\right\}$

[36] $\left.\begin{array}{l} x=\pm4,\ \pm3,\ \pm4\sqrt{-1},\ \pm3\sqrt{-1}; \\ y=\pm3,\ \pm4,\ \pm3\sqrt{-1},\ \pm4\sqrt{-1}. \end{array}\right\}$

[37] $\left.\begin{array}{l} x=11, \\ y=3. \end{array}\right\}$
[38] $\left.\begin{array}{l} x=\pm5,\ \pm2,\ \pm(\pm5\sqrt{-2}\pm\sqrt{370\cdot5})^{\frac{1}{2}}, \\ y=\pm2,\ \pm5,\ \pm(\pm5\sqrt{-2}\mp\sqrt{370\cdot5})^{\frac{1}{2}}. \end{array}\right\}$

[39] $\left.\begin{array}{l} x=\frac{1}{2}\left[b\pm\left\{\pm2\left(\frac{4a^5+b^5}{5b}\right)^{\frac{1}{2}}-b^2\right\}^{\frac{1}{2}}\right], \\ y=\frac{1}{2}\left[b\mp\left\{\pm2\left(\frac{4a^5+b^5}{5b}\right)^{\frac{1}{2}}-b^2\right\}^{\frac{1}{2}}\right]. \end{array}\right\}$

[40] $\left.\begin{array}{l} x=\pm(10)^{\frac{1}{5}}(\cdot5+\cdot1\times5^{\frac{1}{2}}), \\ z=\pm(10)^{\frac{1}{5}}(\cdot5-\cdot1\times5^{\frac{1}{2}}). \end{array}\right\}$

[41] $\left.\begin{array}{l} x=3,\ -2,\ \frac{1}{2}\left\{\pm\left(-\frac{79}{3}\right)^{\frac{1}{2}}+1\right\}, \\ y=2,\ -3,\ \frac{1}{2}\left\{\pm\left(-\frac{79}{3}\right)^{\frac{1}{2}}-1\right\}. \end{array}\right\}$

[42] $\left.\begin{array}{l} x=3,\ \frac{4}{7}(205\cdot8)^{\frac{1}{5}}, \\ y=2,\ -(205\cdot8)^{\frac{1}{5}}. \end{array}\right\}$
[43] $\left.\begin{array}{l} x=0,\ 3(3)^{\frac{1}{3}},\ \frac{2}{3}\left(\frac{98}{3}\right)^{\frac{1}{3}}, \\ y=-9,\ \frac{25}{9}. \end{array}\right\}$

[44] $\left.\begin{array}{l} x=81,\ 16,\ \frac{1}{3}(7+36\sqrt{-22}), \\ y=16,\ 81,\ \frac{1}{3}(7-36\sqrt{-22}). \end{array}\right\}$
[45] $\left.\begin{array}{l} x=\frac{1}{4}(1\pm\sqrt{2}), \\ y=\frac{1}{4}. \end{array}\right\}$

Ex. 42.

[46] $\left.\begin{array}{lll} x=8, & -4, & 8(19\mp 8\sqrt{6}), \\ y=4, & 1, & 8(5\mp 2\sqrt{6}). \end{array}\right\}$

[47] $\left.\begin{array}{lll} x=3, & -2, & \frac{1}{2}\{(13)^{\frac{2}{3}}\pm(-11\sqrt[3]{13})^{\frac{1}{2}}\}, \\ y=-2, & 3, & \frac{1}{2}\{(13)^{\frac{2}{3}}\mp(-11\sqrt[3]{13})^{\frac{1}{2}}\}. \end{array}\right\}$

[48] $x=2,\ -1,\ \pm\frac{1}{2}\left(\frac{1}{\sqrt{-2}}+\sqrt{\frac{3}{2}}\right);\quad y=1,\ \pm\frac{1}{\sqrt{-2}}.$

[49] $\left.\begin{array}{ll} x=\pm(na)^{\frac{1}{2}}, & \frac{1}{2}\{n\pm(n^2\mp 4n\sqrt{ab})^{\frac{1}{2}}\}, \\ y=\pm(nb)^{\frac{1}{2}}, & \frac{1}{2}\{n\mp(n^2\mp 4n\sqrt{ab})^{\frac{1}{2}}\}. \end{array}\right\}$

[50] $\left.\begin{array}{l} x=\pm 4,\ \pm 1,\ \pm 4\sqrt{-1},\ \pm\sqrt{-1}, \\ y=\pm 1,\ \pm 4,\ \pm\sqrt{-1},\ \pm 4\sqrt{-1}. \end{array}\right\}$

[51] $\left.\begin{array}{l} x=\pm 3,\ \pm\sqrt{6},\ \pm\left(\frac{15\pm 3\sqrt{5}}{2}\right)^{\frac{1}{2}}, \\ y=2,\ -1,\ \frac{1}{2}(1\pm\sqrt{-47}),\ \frac{1}{2}(1\pm 3\sqrt{5}). \end{array}\right\}$

[52] $\left.\begin{array}{l} x=\pm 3,\ \pm 2,\ \pm\frac{1}{2}(\sqrt{-27}+1);\ \pm 3\sqrt{-1},\ \pm 2\sqrt{-1}, \\ \quad\pm\frac{1}{2}(\sqrt{27}+\sqrt{-1}): \\ y=\pm 2,\ \pm 3,\ \pm\frac{1}{2}(\sqrt{-27}-1); \\ \quad\pm 2\sqrt{-1},\ \pm 3\sqrt{-1},\ \pm\frac{1}{2}(\sqrt{27}-\sqrt{-1}). \end{array}\right\}$

[53] $\left.\begin{array}{l} x=\pm 2,\ \pm\sqrt{-1},\ \pm\left(\pm\sqrt{-13}+\frac{3}{2}\right)^{\frac{1}{2}}, \\ y=\pm 1,\ \pm 2\sqrt{-1},\ \pm\left(\pm\sqrt{-13}-\frac{3}{2}\right)^{\frac{1}{2}}. \end{array}\right\}$

[54] $x=\frac{1}{2}\pm\left(\frac{25\mp 5\sqrt{41}}{8}\right)^{\frac{1}{2}};\quad y=\pm\left(\frac{25\pm 5\sqrt{41}}{8}\right)^{\frac{1}{2}}.$

[55] $\left.\begin{array}{ll} x=\pm(ac)^{\frac{1}{2}}, & \frac{1}{2}\{a-b+c\pm[(a-b+c)^2-4ac]^{\frac{1}{2}}\}, \\ y=\pm(bc)^{\frac{1}{2}}, & \frac{1}{2}\{b+c-a\mp[(a-b+c)^2-4ac]^{\frac{1}{2}}\}. \end{array}\right\}$

Ex. 42.

[56] $\left.\begin{array}{l} x=5, \\ y=4. \end{array}\right\}$

[57] $\left.\begin{array}{l} x=\frac{3}{2}(19\pm\sqrt{105}),\ \frac{3}{2}(-13\pm\sqrt{-87}), \\ y=\frac{1}{6}(3\pm\sqrt{105}),\ \frac{1}{6}(3\pm\sqrt{-87}). \end{array}\right\}$

[58] $\left.\begin{array}{l} x=\pm\left\{a^{\alpha(\alpha-\beta+\alpha\beta)}.\ b^{\beta(\alpha-\beta+\alpha^2)}\right\}^{\frac{1}{\alpha^2-\beta^2}} \\ y=\pm\left\{a^{\alpha(\alpha-\beta-\beta^2)}.\ b^{\beta(\alpha-\beta-\alpha\beta)}\right\}^{\frac{1}{\alpha^2-\beta^2}} \end{array}\right\}$

[59] $\left.\begin{array}{l} x=(-\frac{1}{2}\pm\frac{1}{6}\sqrt{57})^8, \\ y=(-\frac{1}{2}\pm\frac{1}{6}\sqrt{57})^4. \end{array}\right\}$

[60] $\left.\begin{array}{l} x=1, \\ y=2, \\ z=3. \end{array}\right\}$

[61] $\left.\begin{array}{l} x=3, \\ y=5, \\ z=7. \end{array}\right\}$

[62] $\left.\begin{array}{l} x=3, \\ y=7, \\ z=11, \\ w=20. \end{array}\right\}$

[63] $\left.\begin{array}{l} x=\pm4,\ \pm\frac{10}{3}\sqrt{3}, \\ y=\pm3,\ \pm\frac{1}{3}\sqrt{3}, \\ z=\pm2,\ \mp\frac{8}{3}\sqrt{3}. \end{array}\right\}$

[64] $\left.\begin{array}{l} x=2, \\ y=3, \\ z=6. \end{array}\right\}$

[65] $\left.\begin{array}{l} x=9,\ 1, \\ y=3, \\ z=1,\ 9. \end{array}\right\}$

[66] $\left.\begin{array}{l} x=4, \\ y=6,\ 3, \\ z=3,\ 6. \end{array}\right\}$

VII. Problems in Quadratic Equations.

Ex. 43.

[1] 18, 14. [2] 10, 6. [3] 18, 3. [4] 6, 4.

[5] 8, 9. [6] 6, 9. [7] 12, 5. [8] 8, 7.

[9] 7, 2. [10] 8, 6. [11] 5, 3. [12] 18.

[13] A, 120; B, 80. [14] Sherry £2 a doz.; Claret £3 a doz.

[15] 4550. [16] $\frac{1}{2}$, $\frac{3}{2}$, $\sqrt{2}$. [17] 25, 13, 6.

[18] 14, 10, 2. [19] 15, 12, 10, 7.

Ratio, Proportion and Variation.

Ex. 45.

[1] $\frac{77}{88}$, $\frac{80}{88}$; $\frac{703}{925}$, $\frac{700}{925}$. [4] $a+x:a-x$. [6] $\frac{a+x}{a}$; $\frac{a^2-x^2}{a^3-x^3}$.

[7] 1 : 1. [12] $y=7x$. [13] $xy=\frac{12}{25}(x^2+y^2)$.

Ex. 45.

[14] $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$. [15] $y^2 = 4ax$. [17] 1st, 2 gall.; 2nd, 14 gall.

[18] Diam. £$\frac{mcx^2}{(m+1)a^2}$; ruby £$\frac{cx^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}$.

Arithmetical Progression.

Ex. 46.

[1] 59. [2] -35. [3] 0. [4] $\frac{1}{12}$.

[5] $17\frac{12}{23}$. [6] 400. [7] 25452. [8] -72.

[9] 204. [10] $40\frac{1}{2}$. [11] 0. [12] 407.

[13] $20\frac{5}{12}$. [14] 0. [15] 0. [16] 17.

[17] 133. [18] $\frac{n}{12}(13-7n)$. [19] $\frac{n}{3}(9-n)$. [20] $\frac{n-1}{2}$.

[21] $n\{a^2+x^2-(n-3)ax\}$. [22] $\frac{n}{a+b}\left\{na-\frac{n+1}{2}b\right\}$. [23] 4.

[24] 13 or 7. [25] 19 or 18. [26] 60. [27] 4. [28] 3.

[29] $-\frac{3}{2}$. [30] -1. [31] $-\frac{47}{16}$. [32] 207, 297, 387.

[33] 6, 9, 12, 15. [34] -2, -6, -10, -14.

[35] $\frac{4}{5}$, $\frac{3}{5}$, $\frac{2}{5}$, $\frac{1}{5}$, 0, $-\frac{1}{5}$, $-\frac{2}{5}$, $-\frac{3}{5}$, $-\frac{4}{5}$.

[36] $-\frac{1}{2}$, $\frac{1}{4}$, 1, $1\frac{3}{4}$, $2\frac{1}{2}$, $3\frac{1}{4}$, 4. [37] $n=14$. [38] $n=8$.

[39] $\frac{a\{(r+1)(n-p+1)+m(p-q)\}+b\{(r+1)p-m(p-q)\}}{(r+1)(n+1)}$.

[40] 19. [41] $\frac{1}{3}$; $\frac{1}{2}$; $\frac{n}{12}(3n+1)$. [42] $\frac{3}{2}$; $-\frac{4}{5}$. [43] 4.

[44] $\frac{1}{3}$. [45] pth, $\frac{m+n}{2}$; qth, $m-(m-n)\frac{p}{2q}$.

[47] $\frac{1}{2}nr(nr+1)$. [48] $n(3n-1)\frac{a}{2}+n(n-1)(7n-2)\frac{d}{1.2.3}$.

[49] 25. [50] £135 4*s.*; £5 3*s.* [51] 8. [52] 2, 5, 8, 11.

[53] $\frac{P(r-n)+Q(m-r)}{m-n}$.

Ex. 47.

[1] 1240. [2] 12341. [3] $\frac{n}{2}(6n^2+3n-1)$.

Ex. 47.

[4] $\left\{\frac{n(n+1)}{2}\right\}^2$. [5] 440. [6] 2588.

[7] $3n(n+1)(n+3)$. [8] $-2r(4r+5)$.

[9] $\frac{1}{6}n(n+1)(n+2)$. [10] $\frac{1}{24}n(n+1)(n+2)(n+3.)$

[11] $n=3$. [12] 5, 8, 11, &c.

Geometrical Progression.

Ex. 48.

[1] 80. [2] $\frac{27}{32}$. [3] $\frac{4}{9}$. [4] $208\frac{1}{3}$. [5] $-\frac{448}{243}$.

[6] 9841. [7] $3249\frac{7}{8}$. [8] 41·6622976. [9] $5\frac{394}{729}$.

[10] $39\frac{9}{16}$. [11] $-\frac{1261}{384}$. [12] $\frac{181}{576}$.

[13] $\frac{4}{21}\left\{1-\left(-\frac{3}{4}\right)^n\right\}$. [14] $\frac{3}{25}\left\{1-\left(-\frac{2}{3}\right)^n\right\}$.

[15] $\frac{2^{\frac{3}{2}}}{15}\left\{1-\left(-\sqrt{\frac{5}{2}}\right)^n\right\}(\sqrt{5}-\sqrt{2})$. [16] $\frac{1}{3^{\frac{n-4}{3}}}\cdot\frac{(3^{\frac{n}{3}}-1)}{3^{\frac{1}{3}}-1}$.

[17] 8. [18] $5\frac{2}{5}$. [19] $\frac{8}{21}$. [20] $\frac{5}{7}$. [21] 16.

[22] $\frac{3}{5}$. [23] $10\frac{1}{8}$. [24] $2\frac{7}{9}$. [25] $1\frac{16}{33}$. [26] $\frac{2}{15}$.

[27] $\frac{1-x}{1+x}$. [28] $\frac{f+x}{g+x}$. [29] $\frac{x^2}{x+y}$. [30] $\left(\frac{x}{x+y}\right)\left(\frac{x}{y}\right)^{\frac{1}{2}}$.

[31] $\frac{3a(2a)^{\frac{1}{2}}}{2x\{(3a)^{\frac{1}{2}}-(2x)^{\frac{1}{2}}\}}$. [32] $\frac{1}{36}$; $1\frac{8}{55}$; $\frac{358}{1665}$; $\frac{1}{7}$.

[33] $(100)^{\frac{1}{3}}$, $(100)^{\frac{2}{3}}$; 4, 8, 16, 32; 2, 8, 32; $-1, \frac{3}{2}, -\frac{9}{4}, \frac{27}{8}$.

[35] $\frac{a}{b}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$. [36] 49, 1. [37] $\left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$.

[38] pth term $=(mn)^{\frac{1}{2}}$; qth term $=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

[39] $\frac{1}{b^{2n-2}}\cdot\frac{a^{2n}-b^{2n}}{(a^2-b^2)^2}$. [43] £108, £144, £192, £256.

[44] rth mean $=\frac{S_2^{2n}}{S_1}\left\{\frac{r}{m+1}\left(\frac{S_1}{S_2}\right)^q+\frac{m-r+1}{m+1}\left(\frac{S_1}{S_2}\right)^p\right\}\frac{S_1^2-S_2^2}{S_1^{2n}-S_2^{2n}}$.

Ex. 49.

[1] $\dfrac{1-x}{(1+x)^2}$. [2] 3. [3] 1. [4] $4(2^n-1)-3n$.

[5] $2(2^n-1)-n$. [6] $2(2^n-1)+\dfrac{n}{2}(n+1)$.

[7] $5n-\dfrac{9}{2}\left(1-\dfrac{1}{3^n}\right)$. [8] $2^n(2n-3)+3$.

[9] $\dfrac{2x}{(1-x)^3}$; $\dfrac{2x-(n+1)(n+2)x^{n+1}+2n(n+2)x^{n+2}-n(n+1)x^{n+3}}{(1-x)^3}$.

[10] $\dfrac{2x}{(1-2x)^2}$. [11] $\dfrac{(1+6x)3x}{(1-3x)^2}$. [12] 388.

Harmonical Progression.

Ex. 50.

[1] $\left(1, \dfrac{6}{5}, \dfrac{3}{2}\right)$, $(\infty, -6, -3)$; $\left(\dfrac{15}{19}, \dfrac{15}{16}, \dfrac{15}{13}\right)$, $\left(15, -\dfrac{15}{2}, -3\right)$; $\left(\dfrac{8}{11}, \dfrac{4}{5}, \dfrac{8}{9}\right)$, $\left(\dfrac{8}{5}, 2, \dfrac{8}{3}\right)$. [2] $\dfrac{12}{5}$, 3; $\dfrac{12}{5}$, 3, 4, 6.

[3] 7, $5\frac{8}{11}$, $4\frac{11}{13}$, $4\frac{1}{5}$, $3\frac{12}{17}$, $3\frac{6}{19}$; $1\frac{19}{61}$, $1\frac{19}{21}$, $3\frac{11}{23}$.

[4] $\dfrac{(n+1)xy}{ny+x}$, $\dfrac{(n+1)xy}{(n-1)y+2x}$, $\dfrac{(n+1)xy}{(n-2)y+3x}$, &c., $\dfrac{(n+1)xy}{y+nx}$.

[5] 3. [6] x, y, $\dfrac{xy}{2x-y}$, $\dfrac{xy}{3x-2y}$, &c., $\dfrac{xy}{(n-1)x-(n-2)y}$.

[7] $\dfrac{(m-n)MN}{mN-nM}$. [10] 6, 8, 12. [11] 10, 12, 15. [12] 2, 3, 6.

[13] $\dfrac{b}{a}=\pm 2^{\frac{1}{4}}$, $\dfrac{c}{a}=\pm(1\pm 2^{\frac{1}{2}})^{\frac{1}{2}}$. [15] $y=2(a+b)\div\left\{\left(\dfrac{a}{b}\right)^{\frac{1}{4}}+\left(\dfrac{b}{a}\right)^{\frac{1}{4}}\right\}$.

Piles of Balls and Shells.

Ex. 51.

[1] 8436. [2] 11440. [3] 24395. [4] 2561.
[5] 7580. [6] 26059. [7] 78755. [8] 25707.
[9] 104700. [10] 11110. [11] 123225. [12] 740.
[13] 46. [14] 816, 1496.

Permutations and Combinations.

Ex. 52.

[1] 840, 5040. [2] 11880. [3] 40320. [4] 13860.
[5] 34650; 180; 210; 15120. [6] $n=8$. [7] 10.

Ex. 52.

[8] 7. [9] 3. [11] 210, 210. [12] 252; 56. [13] 495; 9.
[14] 376992; 52360. [15] $n=12$. [16] $n=12$. [17] $n=9$.
[18] 52. [19] 127. [20] $n=6$. [21] 255. [22] 816000.
[23] 27907200. [24] $n=7$. [25] $n=15$, $r=6$. [27] 10.
[28] $r=n$. [29] 2304. [30] $(p+1)(q+1)(r+1)\&c.-1$.

Binomial and Multinomial Theorems.

Ex. 53.

[1] $1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7$;
$1+10x+40x^2+80x^3+80x^4+32x^5$;
$1+3x+\frac{15}{4}x^2+\frac{5}{2}x^3+\frac{15}{16}x^4+\frac{3}{16}x^5+\frac{1}{64}x^6$;
$1+3x+\frac{27}{8}x^2+\frac{27}{16}x^3+\frac{81}{256}x^4$.

[2] $1-18x+135x^2-540x^3+1215x^4-1458x^5+729x^6$;
$1-\frac{5}{2}x+\frac{5}{2}x^2-\frac{5}{4}x^3+\frac{5}{16}x^4-\frac{1}{32}x^5$;
$1+x+\frac{5}{8}x^2+\frac{5}{16}x^3+\frac{35}{256}x^4+\&c.$;
$1-\frac{10}{3}x^2+\frac{20}{3}x^4-\frac{280}{27}x^6+\frac{1120}{81}x^8-\&c.$

[3] $a^9+9a^8x+36a^7x^2+84a^6x^3+126a^5x^4+126a^4x^5+84a^3x^6$
$+36a^2x^7+9ax^8+x^9$;
$a^7-7a^6x+21a^5x^2-35a^4x^3+35a^3x^4-21a^2x^5+7ax^6-x^7$;
$16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$;
$15625-3125x+\frac{3125}{12}x^2-\frac{625}{54}x^3+\frac{125}{432}x^4-\frac{5}{1296}x^5+\frac{1}{46656}x^6$.

[4] $a^{-10}+10a^{-12}x+60a^{-14}x^2+280a^{-16}x^3+1120a^{-18}x^4+\&c.$;
$\frac{a^4}{81}+\frac{8a^5x}{243}+\frac{40a^6x^2}{729}+\frac{160a^7x^3}{2187}+\frac{560a^8x^4}{6561}+\&c.$;
$\frac{1}{c^t}+t\frac{x}{c^{t+1}}+\frac{t(t+1)}{1.2}\cdot\frac{x^2}{c^{t+2}}+\&c.+\frac{t(t+1)\&c.(t+r-1)}{1.2.\&c.r}\cdot\frac{x^r}{c^{t+r}}+\&c.$;
$\frac{1}{a^s}-s\frac{h}{a^{s+1}}+\frac{s(s+1)}{1.2}\frac{h^2}{a^{s+2}}+\&c.+(-1)^r\frac{s(s+1)\&c.(s+r-1)}{1.2.\&c.r}\frac{h^r}{a^{s+r}}\pm\&c.$

[5] $1+x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{8}x^4+\&c.$;

$1-\frac{5}{6}x-\frac{5}{72}x^2-\frac{35}{1296}x^3-\&c.-\frac{5.1.7.13.\&c.(6r-11)}{1.2.3.4.\&c.r}\left(\frac{x}{6}\right)^r-\&c.$;

Ex. 53.

[5] $a^{\frac{3}{4}}+a^{-\frac{1}{4}}x-\frac{1}{6}a^{-\frac{5}{4}}x^2+\frac{5}{54}a^{-\frac{9}{4}}x^3-\&c.$

$$+(-1)^{r-1}.3\frac{1.5.9\ldots(4r-7)}{1.2.3.4\ldots r\times 3^r}a^{\frac{3}{4}-r}x^r\ \&c.;$$

$\left(\frac{2}{3}\right)^{\frac{2}{3}}x^{\frac{2}{3}}-\left(\frac{3}{2}\right)^{\frac{1}{3}}x^{-\frac{1}{3}}y-\frac{3}{8}\left(\frac{3}{2}\right)^{\frac{1}{3}}x^{-\frac{4}{3}}y^2-\&c.$

$$-\frac{2.1.4.7.\&c.(3r-5)}{1.2.3.4.\&c.r\times 2^r}\left(\frac{3}{2}\right)^{r-\frac{2}{3}}x^{\frac{2}{3}-r}y^r-\&c.$$

[6] $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+\&c.+\frac{1.3.5.\&c.(2r-1)}{2.4.6.\&c.(2r)}x^r+\&c.$

$1+\frac{3}{2}x^2+\frac{15}{8}x^4+\frac{35}{16}x^6+\frac{315}{128}x^8+\&c.+\frac{3.5.7.\&c.(2r+1)}{2.4.6.\&c.(2r)}x^{2r}+\&c.$

$a^{-\frac{2}{3}}+\frac{1}{3}a^{-\frac{8}{3}}x^2+\frac{2}{9}a^{-\frac{14}{3}}x^4+\frac{14}{81}a^{-\frac{20}{3}}x^6+\&c.$

$$+\frac{1.4.7.\&c.(3r-2)}{3.6.9.\&c.(3r)}a^{-(2r+\frac{2}{3})}x^{2r}+\&c.$$

$x^{-2}-\frac{2}{5}x^{-7}z^5+\frac{14}{50}x^{-12}z^{10}-\frac{28}{125}x^{-17}z^{15}+\&c.$

$$\pm\frac{2.7.12.\&c.(5r-3)}{1.2.3.\&c.r\times 5^r}x^{-(5r+2)}z^{5r}\mp\&c.$$

[7] $a^{-1}-a^{-\frac{4}{3}}x^{\frac{1}{3}}+a^{-\frac{5}{3}}x^{\frac{2}{3}}-a^{-2}x+a^{-\frac{7}{3}}x^{\frac{4}{3}}-\&c.$

$a^{-2}\left\{1+6\left(\frac{x}{a}\right)^{\frac{1}{3}}+21\left(\frac{x}{a}\right)^{\frac{2}{3}}+56\left(\frac{x}{a}\right)+\&c.+\frac{6.7.\&c.(r+5)}{1.2.\&c.r}\left(\frac{x}{a}\right)^{\frac{r}{3}}+\&c.\right\}.$

$\left(\frac{a^2}{x}\right)^{\frac{1}{3}}\left\{1+\frac{x}{a}+\frac{2}{3}\left(\frac{x}{a}\right)^2+\frac{14}{27}\left(\frac{x}{a}\right)^3+\&c.+\frac{4.7.10.\&c.(3r-2)}{6.9.12.\&c.(3r)}\left(\frac{x}{a}\right)^r+\&c.\right\}.$

[8] $h^7+7h^6k\sqrt{-1}-21h^5k^2-35h^4k^3\sqrt{-1}+35h^3k^4$

$$+21h^2k^5\sqrt{-1}-7hk^6-k^7\sqrt{-1}.$$

$b^8-8b^7y\sqrt{-1}-28b^6y^2+56b^5y^3\sqrt{-1}+70b^4y^4$

$$-56b^3y^5\sqrt{-1}-28b^2y^6+8by^7\sqrt{-1}+y^8.$$

$(-1)^n\Big\{a^{3n}-na^{3n-3}x\sqrt{-1}+\frac{n(n-1)}{1.2}a^{3n-6}x^2$

$$+\frac{n(n-1)(n-2)}{1.2.3}a^{3n-9}x^3\sqrt{-1}-\&c.\Big\}.$$

[9] $1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10};$

$1-6x+15x^2-20x^3+15x^4-6x^5+x^6;$

$a^4-8a^3b+12a^3c+24a^2b^2-72a^2bc+54a^2c^2-32ab^3+144ab^2c$

$$-216abc^2+108ac^3+16b^4-96b^3c+216b^2c^2-216bc^3+81c^4.$$

Ex. 53.

[10] $1+2x+3x^2+4x^3+\&c.$; $\quad 1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\&c.$

[11] $-126a^4$. [12] $-1400000a^{12}$. [13] $-\frac{5}{1024}a^{-\frac{35}{2}}b^{18}$.

[14] $+145$. [15] $210a^2b^2c^3$.

[16] $-\frac{m(m-n)(m-2n)}{1.2.3n^3}\left\{3\frac{bc^2}{a^3}-\frac{m-3n}{n}\cdot\frac{b^3c}{a^4}+\frac{(m-3n)(m-4n)}{4.5.n^2}\cdot\frac{b^5}{a^5}\right\}a^{\frac{m}{n}}$.

[17] $\frac{n(n-1)(n-2)(n-3)(n-4)}{12}$. [18] $\pm\frac{1.6.11.\&c.(5r-4)}{5.10.15.\&c.(5r)}\,\frac{2^r}{(3a)^{r+\frac{1}{5}}}$

[19] $6ab^5+60a^2b^3c+60a^3b^2d+60a^3bc^2+30a^4cd$. [20] -36120.

[21] 4368. [22] $\frac{1}{6}(r+1)(r+2)(r+3)$. [23] 5103000000.

[24] $-\frac{7\times 3^5}{2\times 4^8}\cdot\frac{y^{\frac{5}{3}}}{a^9c^{\frac{7}{6}}x^{\frac{9}{2}}}$; $\quad -\frac{357}{5^6}\cdot\frac{b^5x^{\frac{28}{5}}}{a^{\frac{22}{5}}}\sqrt{-1}$.

[25] $715\frac{(2y)^4}{(3x)^{14}}$; $\quad \frac{60}{2401}\,\frac{t^4}{s^{\frac{23}{7}}}$. [26] $\frac{5.1.3.7.\&c.(4r-9)}{4.8.12.16.\&c.(4r)}\cdot\frac{(3z)^{\frac{r}{2}}}{x^{r-\frac{5}{4}}y^{\frac{r}{2}-\frac{5}{4}}}$.

[27] $\frac{5}{4}$, the 2nd term. [28] $\frac{1.3.5.\&c.(2n-1)}{1.2.3.\&c.n}(2x)^n$; $\quad 73789x^{12}$.

[29] $108073x^{10}$. [30] 55; 35. [32] $\left(\frac{2}{3}\right)^{\frac{1}{2}}$.

[33] $\frac{2-n}{2^{n+1}}$. [34] $\frac{2^{n+1}-1}{n+1}$. [35] 0. [36] $r=\frac{n}{2}-1$.

[37] $n=8$. [38] $n=\frac{7}{3}$. [41] $\frac{1.3.5.\&c.(2n-1)}{1.2.3.\&c.n}(2^n)$.

[42] $\frac{1.3.5.\&c.(2n-1)}{1.2.3.\&c.n}\left(\frac{n}{n+1}\right)2^n$.

Indeterminate Coefficients.

Ex. 54.

[1] $-\frac{2}{x-1}-\frac{10}{x-2}+\frac{18}{x-3}$.

[2] $\frac{a^2+ha+k}{(a-b)(a-c)(x-a)}+\frac{b^2+hb+k}{(b-a)(b-c)(x-b)}+\frac{c^2+hc+k}{(c-a)(c-b)(x-c)}$.

[3] $\frac{1}{6(x+1)}-\frac{1}{2(x-1)}+\frac{4}{3(x-2)}$.

[4] $\frac{1}{4a^3(x-a)}-\frac{1}{4a^3(x+a)}-\frac{1}{2a^2(x^2+a^2)}$.

Ex. 54.

[5] $\frac{10}{1+2x}-\frac{5}{1+x}-\frac{3}{(1+2x)^2}-\frac{1}{(1+x)^2}$.

[6] $\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}+\frac{x-1}{x^2+1}-\frac{1}{(x^2+1)^2}$.

[7] $-\frac{3}{x+2}+\frac{3}{x+3}+\frac{4}{(x+3)^2}$.

[8] $\frac{51}{26(x-5)}-\frac{6x+31}{13(x^2-2x-2)}$.

[9] $\frac{3}{5}-\frac{11}{5^2}x+\frac{7\times 11}{5^3}x^2-\frac{7^2\times 11}{5^4}x^3+\frac{7^3\times 11}{5^5}x^4-$&c.

[10] $1+3x+4x^2+7x^3+11x^4+18x^5+$&c.

[11] $x+(c-a)x^3+(c^2-ac+b-d)x^5+(c^3-ac^2+ad+bc-2cd)x^7+$&c.

[12] $\frac{1}{3}+\frac{1}{9}x-\frac{7}{54}x^2-\frac{95}{324}x^3\pm\ldots+\left(\frac{1}{2^{n-1}}-\frac{1}{2}-\frac{7}{6}\cdot\frac{1}{3^n}\right)x^n+\ldots$

Reversion of Series.

Ex. 55.

[1] $x=-\frac{1}{2}(y-1)+\frac{3}{8}(y-1)^2-\frac{9}{16}(y-1)^3+\frac{135}{128}(y-1)^4-$&c.

[2] $x=\frac{1}{b}(y-a)-\frac{c}{b^3}(y-a)^2+\frac{2c^2}{b^5}(y-a)^3-\frac{5c^3}{b^7}(y-a)^4+$&c.

[3] $x=(y-1)+2(y-1)^2+7(y-1)^3+30(y-1)^4+$&c.

[4] $x=\frac{1}{b}(y-a)-\frac{c}{b^3}(y-a)^2+\frac{2c^2-bd}{b^5}(y-a)^3-\frac{5c^3-5bcd}{b^7}(y-a)^4+$&c.

[5] $x=\frac{1}{2}(y-1)-\frac{1}{2}(y-1)^2+\frac{1}{2}(y-1)^3-\frac{1}{2}(y-1)^4+$&c.

[6] $x=y+\frac{y^2}{1.2}+\frac{y^3}{1.2.3}+\frac{y^4}{1.2.3.4}+$&c.

[7] $x=(y-1)-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\frac{1}{4}(y-1)^4+$&c.

[8] $x=y+\frac{1}{2}\cdot\frac{y^3}{3}+\frac{1.3}{2.4}\cdot\frac{y^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{y^7}{7}+$&c.

[9] $x=y-\frac{y^3}{1.2.3}+\frac{y^5}{1.2.3.4.5}-\frac{y^7}{1.2.3.4.5.6.7}+$&c.

[10] $x=\frac{1}{a}y-\frac{b}{a^3}y^2+\frac{2b^2-ac}{a^5}y^3-\frac{5bc^3-5abc+a^2d}{a^7}y^4+$&c.

[11] $x=\sqrt{3}-\frac{1}{6}y-\frac{\sqrt{3}}{72}y^2-\frac{1}{162}y^3-\frac{35\sqrt{3}}{5184}y^4-$&c.

[12] $x=1+\frac{y}{3n}-\frac{y^3}{3^4n^3}+\frac{y^4}{3^5n^4}-$&c.

Ex. 55.

[13] $x=b+\frac{ay}{3b}-\frac{a^3y^3}{3^4b^5}+\frac{a^4y^4}{3^5b^7}-\&c.$

Summation of Series.

Ex. 56.

[1] 506. [2] $\frac{1}{2}n(6n^2-3n-1)$. [3] 440.

[4] $\frac{1}{3}n(n+1)(4n-1)$. [5] $\frac{1}{12}n(n+1)(n+2)(3n+5)$.

[6] $\frac{1}{4}n(n+1)(n+2)(n+3)$. [7] 44100.

[8] $n^2(2n^2-1)$. [9] 7305.

Ex. 57.

[1] $\frac{n}{n+1}$; 1. [2] $\frac{n}{3(2n+3)}$; $\frac{1}{6}$.

[3] $\frac{n}{3n+1}$; $\frac{1}{3}$. [4] $\frac{n}{3(5n+3)}$; $\frac{1}{15}$.

[5] $\frac{n(n+3)}{4(n+1)(n+2)}$; $\frac{1}{4}$. [6] $\frac{n(3n+5)}{8(3n+1)(3n+4)}$; $\frac{1}{24}$.

[7] $\frac{n(n+1)}{2(2n+1)(2n+3)}$; $\frac{1}{8}$. [8] $\frac{n(n+3)}{10(2n+1)(2n+5)}$; $\frac{1}{40}$.

Ex. 58.

[1] $\frac{n(3n+5)}{4(n+1)(n+2)}$; $\frac{3}{4}$. [2] $\frac{n}{3(4n+3)}$; $\frac{1}{12}$.

[3] $\frac{n(11n^2+48n+49)}{18(n+1)(n+2)(n+3)}$; $\frac{11}{18}$. [4] $1-\frac{1}{n+1}\cdot\frac{1}{2^n}$; 1.

[5] $\frac{n(3n+7)}{2(n+1)(n+2)}$; $\frac{3}{2}$. [6] $1-\frac{n+4}{(n+1)(n+2)}\cdot\frac{1}{2^{n+1}}$; 1.

Ex. 59.

[1] $\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$. [2] $\frac{1+2x-(3n+1)x^n+(3n-2)x^{n+1}}{(1-x)^2}$.

[3] $\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}$.

[4] $\frac{1+6x+x^2-(2n+1)^2x^n+2(4n^2-3)x^{n+1}-(2n-1)^2x^{n+2}}{(1-x)^3}$.

[5] $2(1-x+x^2-x^3+\&c.)-(1+\frac{1}{3}x+\frac{1}{9}x^2+\frac{1}{27}x^3+\&c.)$.

Ex. 60.

[1] 1365. [2] 8694.

[3] $\frac{1}{3}n(n+1)(n+2)$. [4] $\frac{1}{4}n(n+1)(n+2)(n+3)$.

[5] $n^2(2n^2+3)+\frac{5}{3}n(4n^2-1)$. [6] $2n^2(n^2+14)+\frac{10}{3}n(4n^2+5)$.

[7] $\left\{\frac{n(n+1)}{1.2}\right\}^2$. [8] $\frac{n^4}{20}(8n+15)+\frac{n^2}{12}(2n-3)-\frac{n}{15}$.

[9] 44330. [10] 8361. [11] 38760. [12] 305825.

[13] 20; 210; 1540; 8855; 42504.

Interpolation of Series.

Ex. 61.

[1] 82. [2] 3, 7, 15, 30, 55, 93, 147, 220, 315, 435.

[5] 1, $\frac{195}{128}$, $\frac{280}{128}$, $\frac{385}{128}$, 4, $\frac{663}{128}$, &c.

Chances or Probabilities.

Ex. 62.

[1] $\frac{5}{36}$. [2] $\frac{1}{270725}$. [3] $\frac{1}{7}$. [4] $\frac{1}{12}$.

[5] $\frac{1}{2048}$. [6] $\frac{35}{128}$. [7] $\frac{1526}{7776}$. [8] $\frac{6}{29}$.

[9] £1 12*s*. 1·2*d*. [10] $\frac{625}{5184}$. [11] $\frac{1}{6}$. [12] $\frac{C_p^{(m)}.C_q^{(n)}}{C_{p+q}^{(m+n)}}$.

[13] $\frac{60}{253}$. [14] $\frac{7}{16}$ nearly; $\frac{7}{64}$ nearly. [15] $6^{12}:6^6\times5^5\times11:5^{10}\times136$.

[16] $\frac{26}{45}$. [17] $\frac{25}{72}$. [18] 3·8018. [19] 10. [20] $\frac{125}{1296}$.

Scales of Notation.

Ex. 63.

[1] 21101022; 7338. [2] 203116; 6500445.

[3] 212231. [4] 398*e*; 2317; 8751215.

[5] 6*t*12; 25*t*0; 15012980. [6] 1411103040.

[7] 42202772. [8] 1456. [9] *t*4*tee*. [10] *ette*.

[11] 4112; 62*te*. [12] Senary. [13] Nonary. [14] 7.

Ex. 63.

[15] $\frac{19021}{645}$. [16] 1lb., 2lb., 2^2lb., 2^4lb., 2^5lb., 2^7lb., 2^9lb., 2^{10}lb.

[17] Place 1 lb., 3^2 lb., 3^4 lb., 3^5 lb., in one scale, and 3 lb., 3^3 lb. in the other.

Logarithms.

Ex. 64.

[1] 146·0347.	[2] 14041·28.	[3] 23·113327.
[4] 12811·36.	[5] 7001602.	[6] 17385.
[7] ·00250075.	[8] ·000607592.	[9] ·01375935.
[10] ·000001712.	[11] 103·6617.	[12] 1185·57.
[13] ·436859.	[14] 1814·215.	

Ex. 65.

[1] 6·2124.	[2] ·076.	[3] 27·997.
[4] ·0275227.	[5] ·923502.	[6] 8·89.
[7] 3648·24.	[8] ·0590177.	[9] ·0699268.
[10] ·222361.		

Ex. 66.

[1] 1281·1.	[2] 30071·7.	[3] 125·064.
[4] ·0000100612.	[5] ·76570.	[6] 3·35170.
[7] 1·03445.	[8] 1·06470.	[9] 1·688690.
[10] ·097147.		

Ex. 67.

[1] 1·9030.	[2] 15·6757.	[3] 58·44.
[4] 7·57077.	[5] ·08075.	[6] ·0883690.
[7] ·611685.	[8] 2·25996.	[9] ·702946.
[10] 1·0838.	[11] 1·00012.	[12] ·913613.
[13] 0·4254650.	[14] 1728·712.	

Ex. 68.

[1] ·013506.	[2] ·434111.	[3] ·087717.
[4] ·649357.	[5] ·097416.	
[6] 3·20742.	[7] 17·3306.	[8] ·0085161.
[9] ·113888.	[10] ·025776.	[11] 1·2185.

Ex. 69.

[1] 18·342. [2] ·153553. [3] ·00148897.
[4] ·019875. [5] ·000016025. [6] ·94361.
[7] 1·344435. [8] ·0000029979. [9] ·159133.
[10] ·22095. [11] ·1023. [12] £69 2*s*. 4·68*d*.

Ex. 70.

[1] $\log 6 = \cdot 778151$, $\log 15 = 1\cdot 176091$, $\log 5\cdot 4 = \cdot 732393$,
$\log 17\cdot 5 = 1\cdot 243038$, $\log \cdot 875 = \bar{1}\cdot 942008$, $\log 6860 = 3\cdot 836324$.

[2] $\log 2 = \cdot 301030$, $\log 3 = \cdot 477121$, $\log \cdot 16 = \bar{1}\cdot 204120$.
$\log 450 = 2\cdot 653212$, $\log \cdot 075 = \bar{2}\cdot 875061$, $\log 3\cdot 75 = \cdot 574031$.

[3] $5\cdot\dot{3}$.

Ex. 71.

[1] $x = 1\cdot 537$. [2] $x = 9\cdot 5868$. [3] $x = \cdot 5849$.
[4] $x = 17\cdot 917$. [5] $x = \cdot 37166$. [6] $x = 2\cdot 342$.
[7] $x = 1\cdot 312$. [8] $x = 9\cdot 673$.

[9] $x = \dfrac{\log c - d\log a}{b\log a}$. [10] $x = \dfrac{\log c}{m\log a + n\log b}$.

[11] $x = 1\cdot 242$. [12] $x = 3\cdot 127$. [13] $x = 1\cdot 6624$, $y = 1\cdot 2764$.
[14] $x = \cdot 98121$, $y = \cdot 33920$. [15] $x = 4\cdot 2818$, $y = 3\cdot 0584$.
[16] $x = 3\cdot 5510$, $y = 1\cdot 4204$. [17] $x = -1\cdot 3533$, $y = 4\cdot 0229$.

[18] $x = \left(\dfrac{m^2\log c}{m^2\log a + n^2\log b}\right)^{\frac{1}{2}}$, $y = \left(\dfrac{n^2\log c}{m^2\log a + n^2\log b}\right)^{\frac{1}{2}}$.

[19] $x = 2\cdot 25$, $y = 3\cdot 375$. [20] $x = \dfrac{\log(a-b)}{\log(a+b)}$.

[21] $x = \dfrac{\log(\sqrt{2}+1)}{2\log a}$. [22] $x = \dfrac{\log\{c \pm (1+c^2)^{\frac{1}{2}}\}}{\log a}$. [23] $x = \left(\dfrac{\log n}{\log a}\right)^{\frac{1}{2}}$.

Interest and Annuities.

Ex. 72.

[1] £889 4*s*. [2] £3223 4*s*. 7*d*. [3] £613 18*s*. 4*d*.
[4] £497 11*s*. 9*d*. [5] £13 12*s*. $4\frac{3}{4}$*d*. [6] £12 4*s*. 11*d*.
[7] 47·303 yrs. [8] 36·894 yrs. [9] $6\frac{1}{2}$ yrs. nearly
[10] £12 7*s*. 4·8*d*. [11] £70 13*s*. 5*d*.
[12] A's share £1274 7*s*. $1\frac{1}{2}$*d*.; B's, £1178 4*s*. $3\frac{1}{2}$*d*.; C's, £1047 8*s*. 7*d*.
[13] $n = 16\cdot 67$ &c. yrs. [14] £12547. [16] £3817 3*s*. 7*d*.
[17] £6 16*s*. $6\frac{1}{2}$*d*. [18] £17 14*s*. $2\frac{1}{2}$*d*. [19] £120 14*s*. $7\frac{1}{4}$*d*.
[20] After 12 yrs. [21] £218 13*s*. $2\frac{1}{2}$*d*. [22] £34 1*s*. $3\frac{1}{4}$*d*.
[23] £933 8*s*. $9\frac{1}{2}$*d*. present worth; £4 1*s*. 10*d*. per cent.

THEORY OF EQUATIONS.

Ex. 1.

[1] $x^2-2x-15=0$. [2] $x^3-8x^2+x+42=0$.

[3] $x^4+4x^3-7x^2-10x=0$. [4] $225x^4-436x^2+64=0$.

[5] $9x^5-6x^4-439x^3+294x^2-98x=0$.

[6] $x^4-10x^3-19x^2+480x-1392=0$.

[7] $x^4+x^2-6=0$. [8] $x^4-2x^3+(1+c-a^2)x^2-2cx+c(1-a^2)=$

[9] $29x^2$. [10] $28x^3$. [11] 7, 6, 4, 2.

[12] $X' =5x^4-40x^3+87x^2-20x-62$; and $X' =5x^4-3px^2-2qx$;

$X'' =20x^3-120x^2+174x-20$; $X'' =20x^3-6px-2q$;

$X''' =60x^2-240x+174$; $X''' =60x^2-6p$;

$X^{(iv)}=120x-240$; $X^{(iv)}=120x$;

$X^{(v)}=120$. $X^{(v)}=120$.

Ex. 2.

[1] $x^3-15x^2+72x-14=0$. [2] 3, 7.

[3] $x^3+6x^2-31x-120=0$. [4] $x^4+x^3-16x^2+4x-80=0$.

[5] $x^8+2x^7-7x^6-4x^5-12x^4-4x^3-7x^2+2x+1=0$.

Ex. 3.

[1] $3\pm\sqrt{2}$, 5. [2] $-\frac{1}{2}(3\pm\sqrt{-31})$, 4, -1.

[3] 3, $\pm\sqrt{2}$, 2, 5. [4] 2, 2, 3.

[5] 2, 2, 1. [6] 2, 2, -3. [7] $\frac{1}{2}$, $\frac{1}{2}$, $\frac{-1\pm\sqrt{-2}}{2}$.

[8] -1, -1, -1, -10. [9] 1, 1, 6, 6. [10] 3, 3, 3, 2, 2.

[11] 1, 3, 5. [12] -2, 1, 4. [13] 1, 2, 3, 4. [14] 5, 2, -1, -4.

[15] 1, 3, 9. [16] 2, 4, 8. [17] 2, 6, 18.

[18] If α, $\alpha\rho$, $\alpha\rho^2$, $\alpha\rho^3$ be the roots; then ρ is found from

$$\rho+\frac{1}{\rho}=\frac{1}{2}\{\pm\left(\frac{4pq+r}{r}\right)^{\frac{1}{2}}-1\};\text{ and } \alpha \text{ from } \alpha^2\rho^3=\pm s^{\frac{1}{2}}.$$

[19] 1, 2, 4, 8. [20] 3, $\frac{1}{3}$, 2, $\frac{1}{2}$.

Ex. 4.

[1] $y^3+12y^2+9y+24=0$.

[2] $y^3-14y^2+11y-75=0$.

[3] $y^4-25y^3+375y^2-1260y-11700=0$.

Ex. 4.

[4] $y^4-48y^2-81y+90=0.$

[5] $y^4+63y^2-108y+243=0.$

[6] $y^4+10y^3-875y-625=0.$

[7] $4y^3-6y^2+4y+5=0.$

[8] $y^3-6y^2+11y-3=0.$

[9] $y^3+9y^2-90=0.$

[10] $y^4+2y^3-152y^2-1153y-2331=0.$

[11] $y^5+12y^4+41y^3+26y^2-160y-336=0.$

[12] $y^4+12y^3+27y^2-68y-84=0.$

[13] $y^4-26y^3+100y^2-103y+7=0.$

[14] $y^5-6y^4+7{\cdot}4y^3+7{\cdot}92y^2-17{\cdot}872y-{\cdot}79232=0.$

[15] $2y^4+8y^3-y^2-8y-20=0.$

[16] $19y^4+206y^3+793y^2+1232y+580=0.$

Ex. 5.

[1] $y^3-5y-4=0.$ [2] $y^3-12y-11=0.$

[3] $y^3+27=0.$ [4] $y^3-31y+74=0.$

[5] $y^3-8y-15=0.$ [6] $y^3-\frac{19}{3}y-\frac{79}{27}=0.$

[7] $y^3-\frac{7}{3}y-\frac{169}{27}=0.$ [8] $y^4-23y^2+59y-52=0.$

[9] $y^4-\frac{27}{8}y^2+\frac{13}{8}y-\frac{819}{256}=0.$ [10] $y^4-37y^2-123y-110=0.$

[11] $27y^3-152=0.$ [12] $y^5-3{\cdot}6y^3-2{\cdot}68y^2+4{\cdot}456y+3{\cdot}99104=0.$

[13] $y^3+y^2-\frac{139}{27}=0$; or $y^3-y^2-5=0.$

[14] $y^3+3y^2-20=0$; or $y^3-3y^2-16=0.$

[15] $y^3-y^2=0$; or $y^3+y^2-\frac{4}{27}=0.$

[16] $y^4-22y^3+63y-44=0$; or $y^4+22y^3-2599y-13992=0.$

[17] $27y^4+67y-59=0.$

[18] $2y^5-\frac{7}{2}y^2+\frac{23}{8}y+\frac{3}{8}=0.$

Symmetrical Functions.

Ex. 6.

[1] $S_6=5.$ [2] $S_6=795$; $S_{-2}=\frac{85}{36}.$

Ex. 6.

[3] $S_5 = -2849$; $S_{-3} = \frac{31159}{27000}$.

Ex. 7.

[1] $x^3 - 40x - 39 = 0$. [2] $x^3 - 12x - 16 = 0$.

[3] $x^3 - 20x^2 - 56x - 100 = 0$. [4] $x^3 - 5x^2 + 3x + 1 = 0$.

Ex. 8.

[1] $x^3 - 2px^2 + (p^2 + q)x - (pq - r) = 0$.

[2] $x^3 - qx^2 + prx - r^2 = 0$.

[3] $x^3 - (p^2 - 2q)x^2 + (q^2 - 2pr)x - r^2 = 0$.

[4] $x^3 - 2(p^2 - 2q)x^2 + (p^4 - 4p^2q + 5q^2 - 2pr)x$
$- (p^2q^2 - 2p^3r + 4pqr - 2q^3 - r^2) = 0$.

[5] $r^2x^3 - (q^2 - 2pr)x^2 + (p^2 - 2q)x - 1 = 0$.

[6] $r^2x^3 - (pq - 3r)rx^2 + (p^3r - 5pqr + 3r^2 + q^3)x$
$- (p^2q^2 - 2p^3r + 4pqr - 2q^3 - r^2) = 0$.

[7] $x^3 - (p^3 - 3pq + 3r)x^2 + (3r^2 - 3pqr + q^3)x - r^3 = 0$.

Reciprocal or Recurring Equations.

Ex. 9.

[1] $3 \pm 2\sqrt{2}$; $2 \pm \sqrt{3}$. [2] $\frac{1}{2}(-5 \pm \sqrt{21})$; $\pm \sqrt{-1}$.

[3] $2, \frac{1}{2}$; $\pm \sqrt{-1}$. [4] $-1, \frac{1}{2}\{3 \pm 2\sqrt{-1} \pm (1 \pm 12\sqrt{-1})^{\frac{1}{2}}\}$.

[5] $-1, \frac{1}{4}\{9 \pm \sqrt{53} \pm (118 \pm 18\sqrt{53})^{\frac{1}{2}}\}$. [6] $1, 2, \frac{1}{2}, 2 \pm \sqrt{3}$.

[7] $1, \frac{1}{4}\{5 \pm \sqrt{61} \pm (70 \pm 10\sqrt{61})^{\frac{1}{2}}\}$.

[8] $x = a \pm (a^2 - 1)^{\frac{1}{2}}$, where $a = 2{\cdot}11$; $-1{\cdot}202$; ${\cdot}5914$.

[9] $x = \{(a+1)^{\frac{1}{2}} - a^{\frac{1}{2}}\}^{\frac{1}{2}}.\{(a-1)^{\frac{1}{2}} + a^{\frac{1}{2}}\}^{\frac{1}{2}}$.

[10] -1, $\pm \sqrt{-1}$, $\frac{1}{2}\{1 \pm \sqrt{-3}\}$, $\frac{1}{2}(3 \pm \sqrt{5})$, $-2 \pm \sqrt{3}$.

[11] ± 1, $\pm \left\{\frac{1 \pm \sqrt{-3}}{2}\right\}^{\frac{1}{2}}$, $\pm \left\{\frac{1 \pm \sqrt{-3}}{2}\right\}^{\frac{1}{3}}$.

[12] $1, 1, \frac{1}{4}\{1 \pm \sqrt{-15}\}$. [13] $2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\{1 \pm \sqrt{-3}\}$.

Ex. 9.

[14] $\frac{1}{2}\{\pm\sqrt{2}\pm\sqrt{-2}\}$; ± 1, $\pm\sqrt{-1}$: ± 1, $\pm\left(\frac{1\pm\sqrt{-3}}{2}\right)$:

-1, $\cos(n\times 20^\circ)\pm\sqrt{-1}\sin(n\times 20^\circ)$, where n is 1, 3, 5, 7 successively: $\pm a$, $\pm a\left(\frac{1\pm\sqrt{-3}}{2}\right)$, $\pm a\sqrt{-1}$, $\frac{a}{2}\{\pm\sqrt{3}\pm\sqrt{-1}\}$.

[15] 1, -1, -1, $2\pm\sqrt{3}$.

Equal Roots.

Ex. 10.

[1] $3, 3, -4$.

[2] $2, 2, -1, -3$.

[3] $1, 1, 1, 3$.

[4] $3\pm\sqrt{2}$, $3\pm\sqrt{2}$.

[5] $3, 3, \pm\sqrt{-\frac{1}{2}}$.

[6] $\frac{1}{2}, \frac{1}{2}, \pm\sqrt{2}$.

[7] $2, 2, \frac{2}{3}, -\frac{1}{2}$.

[8] 3, $2\pm\sqrt{2}$, $2\pm\sqrt{2}$.

[9] 1, $\pm\sqrt{-2}$, $\pm\sqrt{-2}$.

[10] $\pm\sqrt{3}$, $\pm\sqrt{3}$, $1\pm\sqrt{-1}$.

[11] $1, 1, 1, -1, -1, -4$.

[12] $3, 3, 3, 2, 2, -1, -1, -4$.

Ex. 11.

[1] $3, 5, 15$.

[2] $2, 3, 6$.

[3] $1, \frac{1}{2}, \frac{1}{3}$.

[4] $-\frac{1}{2}, 1, \frac{1}{4}$.

[5] $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.

[6] 1, $2\pm\sqrt{-8}$; 1, $\frac{1}{2}(1\pm\sqrt{65})$.

[7] 1, $2\pm\sqrt{-5}$; 1, $\frac{1}{2}\{1\pm\sqrt{-11}\}$.

[8] 2, 1, $\pm\sqrt{-2}$.

[9] 5, 6, $-6\pm 2\sqrt{-1}$.

[10] $6, 4, 7$.

Cubic Equations.

Ex. 12.

[1] $3{\cdot}591$.

[2] -4, $2\pm\sqrt{-3}$.

[3] $4^{\frac{1}{3}}-2^{\frac{1}{3}}$, or $\cdot 3274\dot{8}$ &c.

[4] 3, $\frac{1}{2}\{-3\pm\sqrt{-15}\}$.

[5] 5, $2\pm\sqrt{-1}$.

[6] 1, $-2\pm 3\sqrt{-1}$.

[7] 2, $2\pm\sqrt{-1}$.

[8] 6, $2\pm\sqrt{-3}$.

[9] $2, -4, -4$.

[10] 4, $\frac{1}{2}\{-1\pm\sqrt{21}\}$.

Ex. 12.

[11] 8·577, ·776, −1·353. [12] 3, 4, −1.

[13] 2·88879, −2·7639, −·12509. [14] 5, 3, −8.

[15] ·801247. [16] 4·225239. [17] 5, 6·645751, 1·354249.

[18] 8·3066. [19] $\frac{1}{4}$. [20] 7^n.

Biquadratic Equations.

Ex. 13.

[1] $4, -1, -\frac{1}{2}\{3 \pm \sqrt{-31}\}$. [2] $6, -1, -2, -3$.

[3] $1 \pm \sqrt{2}, -1 \pm 2\sqrt{-1}$. [4] $4, 2, -1 \pm \sqrt{-3}$.

[5] $3, 3, 1, -1$. [6] $\pm 1, -4 \pm \sqrt{6}$.

[7] $3, -1, 3 \pm \sqrt{30}$. [8] $1, 2, -2, -3$.

[9] $6, 2, 6 \pm 2\sqrt{-1}$. [10] $\pm 7, \frac{1}{3}(1 \pm \sqrt{-1})$.

Limits of Roots.

Ex. 14.

[1] 4. [2] 3. [3] 3.

[4] 4. [5] 5. [6] 2.

[7] 3. [8] 8. [9] 1.

[10] Not more than two positive roots, and one negative.

Rational Roots.

Ex. 15.

[1] $6, \frac{1}{2}\{3 \pm \sqrt{-7}\}$. [2] $2, 2, -2$.

[3] $-5, 1 \pm \sqrt{-1}$. [4] $3, 6, -4$.

[5] 2. [6] $\frac{3}{2}, \pm\sqrt{-1}$.

[7] $\pm 4, \frac{1}{2}\{1 \pm \sqrt{-11}\}$. [8] $\frac{2}{3}, \pm\sqrt{2}$.

[9] $\frac{5}{2}, \frac{1}{8}\{3 \pm \sqrt{41}\}$. [10] $\frac{3}{2}, \frac{3}{2}, 3(1 \pm \sqrt{2})$.

[11] $2, 2, \frac{2}{3}, -\frac{1}{2}$. [12] $\frac{2}{3}, \frac{3}{2}, \frac{1}{2}\{-11 \pm \sqrt{93}\}$.

STURM'S THEOREM.

N.B. The formula $\{a, b\}$ *denotes that one root lies between* a, b.

Ex. 16.

[1] $\{-4, -3\}$; two roots imaginary.
[2] $\{-2, -1\}$; $\{1, 2\}$; $\{3, 4\}$.
[3] $\{2, 3\}$; two roots imaginary.
[4] -2, $\{-2\frac{1}{2}, -2\frac{1}{4}\}$; $\{4, 5\}$.
[5] $\{-5, -4\}$; $\{-2, -1\}$; $\{5, 6\}$.
[6] $\{-3, -2\}$; $\{-2, -1\}$; $\{3, 4\}$.
[7] $\{-10, -9\}$; $\{3, 4\}$; $\{3, 4\}$.
[8] $\{11, 12\}$; two roots imaginary.
[9] $\{2, 3\}$; $\{3, 4\}$; two roots imaginary.
[10] The roots are $7, -2, -2, -3$.
[11] $\{-2, -1\}$; $\{0, 1\}$; $\{9, 9\frac{1}{2}\}$; $\{9, 9\frac{1}{2}\}$.
[12] $\{-2, -1\}$; $\{10, 11\}$; two roots imaginary.
[13] $5, -3, 2\pm(-3)^{\frac{1}{2}}$.
[14] $\{-2, -1\}$; $\{-1, 0\}$; $\{1, 2\}$; $\{1, 2\}$.
[15] $\{-6, -5\}$; $\{-1, 0\}$; $\{4, 5\}$; two roots imaginary.
[16] $\{1, 2\}$; two pairs of imaginary roots.
[17] $\{-5, -4\}$; two pairs of imaginary roots.
[18] $\{-4, -3\}$; $\{-1, 0\}$; $\{-1, 0\}$; $\{0, 1\}$; $\{3, 4\}$.
[19] $\{-2, -1\}$; $\{1, 2\}$; $\{1, 2\}$; two pairs of imaginary roots.
[20] $\{-14, -13\}$; $\{-10, -9\}$; $\{-6, -5\}$; $\{1, 2\}$; $\{1, 2\}$; $\{1, 2\}$.

APPROXIMATION.

Ex. 17.

[1] 2·09455. [2] 2·4908. [3] 2·7147.
[4] 1·6920 or 1·3568. [5] 1·7837. [6] $-5·5975$.
[7] 2·4573. [8] 2·04727. [9] $-4·00317$.
[10] 2·5293. [11] ·63647. [12] 4·5463.
[13] 7·33555403. [14] 3·3548487. [15] 4·4641016.
[16] ·6386058033. [17] $\frac{3}{1}, \frac{7}{2}, \frac{185}{53}, \frac{192}{55}$.
[18] $(x-a)(x-b)=p^2$, or $(x-a)(x-c)=n^2$, or $(x-b)(x-c)=m^2$.

MENSURATION.

Ex. 1.

[1] 23 A. 2 R. 34·56 P. [2] 1 A. 1 R. 6 P. $10\frac{3}{4}$ Yd. [3] $22\frac{1}{3}$ Ft.
[4] 218·895 Yd. [5] $396\frac{5}{12}$ Ft. [6] 2 R. 21 P. $7\frac{1}{2}$ Yd.
[7] £111 0*s.* 7·7*d.* [8] 57 Yd. [9] 220 Yd.
[10] 15 chains 75 links. [11] 346·4; 20. [12] 31·5; 952·56.
[13] 53° 7′ 48″. [14] $82\frac{2}{3}$ Yd.

Ex. 2.

[1] 31 Ft. [2] 105·6 Yd. [3] 67·658 Ft.
[4] 140·296 Ft. [5] 10A.3R.8P.$12\frac{1}{2}$Yd. [6] 38·411 A.
[7] 83·53 Yd. [8] 3489·224. [9] 236·4 Yd.
[10] 46·188 Ft. [11] £5 11*s.* $6\frac{3}{4}$*d.* [12] 19·065 Yd.
[13] 50° 10′ 28″.

Ex. 3.

[1] 22·1354 Ft. [2] $885\frac{1}{3}$ Ft. [3] 3844·323.
[4] 55·59; 9·064 Ft. [5] 327·066. [6] 22·777.
[7] $128\frac{1}{4}$. [8] 11A. 1R. 7·68P. [9] 41·4622 A.
[10] 52330·3. [11] 17A. 2R. 21P. [12] 27·401 A.

Ex. 4.

[1] 387·11 Ft. [2] 2338·272 Ft. [3] 7358·7 Ft.
[4] 1086·4 Ft. [5] 36·495 Yd. [6] 15·8303 Yd.
[7] 3·9656 Yd. [8] 9·4507 Yd. [9] 2·06457; 2·55195.
[10] 26·216; 29·097. [11] 34·057; 35·494.
[12] Pentagon; 58·778, 72·654 : Octagon; 38·268, 41·421 : Dodecagon; 25·8819, 26·7949.
[13] 416·526. [14] 315·823.

Ex. 5.

[1] 15·708 Ft. [2] 1·5915 chains. [3] 113·0976 Ft.
[4] 50·93 A. [5] 39·25 Yd. [6] £2 9*s.* $9\frac{3}{4}$*d.*
[7] 22·42 Ft. [8] 157·08 Ft. [9] 1005·31 Ft.
[10] 26·7146. [11] $6\frac{1}{2}$*d.* nearly.
[12] 6·6061, 8·6093, 20·7846. [13] 4·633.
[14] 53·088. [15] 32·175. [16] 14·69 In.

Ex. 6.

[1] 88·802 Ft. [2] 315 Yd. [3] 72·526 Ft.
[4] 92·59. [5] 185·088. [6] 132·6.
[7] 319·70. [8] 5 Ft. [9] 91° 40′ 22″·8.
[10] 48·072 Yd.

Ex. 7.

[1] 1776·74 In. [2] 33·53. [3] 8·1066.
[4] 509·38. [5] 192·30. [6] 4229·31.
[7] 441·93. [8] 164·39. [9] 35004·3 Ft.
[10] 89·07. [11] Each area $=\frac{\pi}{2}-1$.
[12] 56 Yd. [13] 801·1 Ft. [14] 162·03.

Ex. 8.

[1] 162 Sq. In.; $121\frac{1}{2}$ Cub. In. [2] 284·924. [3] 897·42.
[4] $932\frac{4}{7}$. [5] $2666\frac{2}{3}$. [6] $16\frac{4}{7}$ Ft.

Ex. 9.

[1] $10\frac{1}{3}$ Sq. Ft.; $\frac{5}{6}$ Cub. Ft. [2] 2880 Ft.
[3] 8·78158 Ft. [4] Surface = 177·0139; solid content = 83·0266.

Ex. 10.

[1] 260·94 Sq. Ft.; 130·98 Cub. Ft. [2] 760.
[3] 744 Sq. Ft.; 1164·924 Cub. Ft. [4] 76·759 Ft.
[5] 37000. [6] 76·737 Ft.; 54·126 Ft. [7] 589·068 Ft.; 418·29 Ft.

Ex. 11.

[1] 251·328 Sq. Ft. [2] 127·2348 Ft.
[3] 14·235 Ft. [4] £6 13*s.* $5\frac{1}{2}$*d.*

Ex. 12.

[1] 254·469 Sq. Ft.; 381·704 Cub. Ft. [2] 1538·8.
[3] $r=\frac{1}{2}\sqrt{6}$; $c=\pi\sqrt{6}$. [4] £237 10*s.* 1*d.*
[5] $\frac{3}{800}$ nearly. [6] 213·629 Sq. Ft.
[7] 1256·64 Ft.; 5236 Ft. [8] 144·925 Ft.; 208·59 Ft.
[9] 26·272 Cub. In. [10] 10*s.* $2\frac{3}{4}$*d.* [11] 7369·18 lb.

Ex. 12.

[12] $h^3-3rh^2+r^3=0$, would give the height h, r being the radius.

[13] 83 : 518 : 399 nearly. [14] 502·65; 1977·1.

[15] $\frac{1}{4}$ (surface); $\frac{5}{32}$ (volume). [16] $\frac{1}{2}$ rad.; $\frac{2}{27}$ (volume).

[17] 5·03167 miles.

Ex. 13.

[1] 210·488 Sq. Ft. [2] 19·24 Ft. [3] 56·83 Ft.

[4] 1187·12 Ft. [5] 199·93 Sq. Ft.; 447·15 Cub. Ft.

[6] 144 Sq. Ft.; 78·83 Cub. Ft. [7] 9·253 Cub. In.; 29·966.

[8] 577·61 lb. [9] π Cub. Ft.

Ex. 14.

[1] 5·289. [2] £13 10*s*. 4*d*. [3] £225 6*s*. 9*d*.

[4] £29 7*s*. 6*d*. [5] £28 4*s*. [6] ·20737 in.

[7] £2 8*s*. 3*d*.

Ex. 15.

[1] 243 lb. [2] 11·246 in. [3] 72·14 lb.

[4] 5·8 in. [5] 5·78 lb. [6] 2·65 in.

[8] 172·8 lb. [9] 14·42 in. [10] 12·72 lb.

[11] 8·3 in. [12] 94·24 lb. [13] 10·61 in.

[14] 4·46 in. [15] 422·23 lb. [16] 17·68 in.

[17] 11·588 in. [18] 1·074 in.

TRIGONOMETRY.

Ex. 1.

[1] $16^g\,74^{\backprime}\,59^{\backprime\backprime}\cdot876$; $125^g\,56^{\backprime}\,63^{\backprime\backprime}\cdot58$; $87^g\,60^{\backprime}\,53^{\backprime\backprime}\cdot39$.

[2] $55°\,44'\,53''$; $-(69°\,18'\,0''\cdot7)$; $-(135°\,31'\,28'')$.

[3] $125°\,49'\,4''$; $35°\,22'\,19''$; $-(134°\,10'\,17'')$.

[4] ·99484; 1·0035; ·31416; ·94248; ·3927.

[5] $57°\,17'\,44''\cdot8$; $28°\,38'\,52''\cdot4$; $85°\,56'\,37''\cdot2$;
$19°\,5'\,54''\cdot9$; $38°\,11'\,49''\cdot8$; $47°\,44'\,47''\cdot2$.

[6] $\sin A=\frac{4}{5}$; $\cot A=\frac{3}{4}$; $\text{chd } A=\sqrt{\frac{4}{5}}$.

Ex. 1.

[7] $\sin 3A = 1$; $\cos 4A = -\frac{1}{2}$; $\tan 2A = \sqrt{3}$.

[8] $\operatorname{cosec} 2A = \frac{25}{24}$; $\operatorname{versin} A = \frac{1}{5}$. [9] $\sin A = \frac{1}{2}$.

Ex. 3.

[1] $\frac{1}{2}\{2-(2+\sqrt{3})^{\frac{1}{2}}\}^{\frac{1}{2}}$; $\frac{1}{4}\{(3+\sqrt{5})^{\frac{1}{2}}-(5-\sqrt{5})^{\frac{1}{2}}\}$;

$\frac{1}{2}(2-\sqrt{2})^{\frac{1}{2}}$; $\frac{1}{4}\{(5+\sqrt{5})^{\frac{1}{2}}-(3-\sqrt{5})^{\frac{1}{2}}\}$.

[2] $\frac{1}{8}\{\sqrt{5}-1+(30+6\sqrt{5})^{\frac{1}{2}}\}$; $\frac{1}{2}\{2+(2+\sqrt{2})^{\frac{1}{2}}\}^{\frac{1}{2}}$;

$\frac{1}{2}\{2+(2-\sqrt{2})^{\frac{1}{2}}\}^{\frac{1}{2}}$; $-\frac{1}{4}\{(3+\sqrt{5})^{\frac{1}{2}}+(5-\sqrt{5})^{\frac{1}{2}}\}$.

[3] $\sqrt{5}+1-(5+2\sqrt{5})^{\frac{1}{2}}$; $\sqrt{2}-1$;
$\sqrt{6}-\sqrt{2}-(2-\sqrt{3})$; $-(2-\sqrt{3})$.

[4] $(5+2\sqrt{5})^{\frac{1}{2}}$; $2-\sqrt{3}$; 1; $-\sqrt{3}$.

[5] $(4-2\sqrt{2})^{\frac{1}{2}}$; $\left\{\frac{1}{5}(10+2\sqrt{5})\right\}^{\frac{1}{2}}$; $-\sqrt{2}$; $-(\sqrt{6}-\sqrt{2})$.

[6] $\frac{2}{3}\sqrt{3}$; $\left\{\frac{1}{5}(10-2\sqrt{5})\right\}^{\frac{1}{2}}$; $-\left\{\frac{1}{5}(10+2\sqrt{5})\right\}^{\frac{1}{2}}$;

$(10-\sqrt{5})^{\frac{1}{2}}+\frac{1}{\sqrt{2}}(3-\sqrt{5})$.

[7] $\frac{1}{2}\{2-(2+\sqrt{3})^{\frac{1}{2}}\}$; $\frac{1}{2}\{2-(2-\sqrt{2})^{\frac{1}{2}}\}$; $\frac{3}{2}$; $1-\frac{1}{4}(10+2\sqrt{5})^{\frac{1}{2}}$.

[8] $\frac{1}{2}(\sqrt{5}-1)$; $(2-\sqrt{2})^{\frac{1}{2}}$; $\sqrt{3}$; $\frac{1}{2}(10-2\sqrt{5})^{\frac{1}{2}}$.

Ex. 8.

[1] $0°$; $60°$. [2] $0°$; $60°$. [3] $0°$; $30°$.
[4] $45°$; $90°$; $135°$. [5] $15°$; $30°$; $45°$; $75°$.
[6] $18°$; $54°$; $180°\pm 18°$; $180°\pm 54°$. [7] $45°$; $\tan^{-1}(3)$.
[8] $60°$; $120°$. [9] $15°$; $75°$. [10] $18°$; $234°$.
[11] $45°$; $120°$. [12] $18°$; $54°$. [13] $45°$; $135°$; $225°$; $315°$.

Ex. 9.

[1] $45°$. [2] $35°$. [3] $15°$; $60°$. [4] $45°$.

Ex. 9.

[5] $\frac{1}{2}\cos^{-1}(2\cos^2 2\alpha)$. [6] $\cos^{-1}(1 \pm \cos\alpha)$.

[7] 2α; $\pi - 2\alpha$. [8] $\cos(x+\alpha) = \pm(n+1)^{\frac{1}{2}}\cos\alpha$.

[9] $\frac{1}{4}\{\alpha + \sin^{-1}(3\sin\alpha)\}$. [10] $x = 36°$; $108°$.

Ex. 11.

[2] $x = \pm(\sqrt{3}+1)$. [3] $x = \pm ab$.

[4] $x = \frac{1}{2}$; -1. [5] $\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\frac{1}{2}u$. [6] a.

[7] $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$. [8] $\frac{\sin\alpha\sin\phi}{1-\sin\alpha\cos\phi}$.

[9] $\frac{2\sin^2\frac{\alpha}{2}\tan\phi}{1+\cos\alpha\tan^2\phi}$. [10] $\frac{1}{m} = \frac{1}{2}(\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta)^{\frac{3}{2}}$.

Ex. 12.

[1] $(mn)^{\frac{2}{3}}(m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1$. [2] $\cos 2\phi = \frac{a^2+c^2-b^2}{2ac}$.

[3] $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\gamma = \tan\frac{1}{2}\beta$.

[4] $(m-b)(n-b)a^2 = (m-a)(n-a)b^2$.

[5] $a^2\sin\theta = b^2\sin\phi$. [6] $\cot 2\omega = \cot 2\alpha + \frac{\mu R^2}{V^2}\operatorname{cosec} 2\alpha$.

Properties of Plane Figures.

Ex. 15.

[7] $a = p\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$; $b = p\frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}C}$; $c = p\frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}A\cos\frac{1}{2}B}$.

[14] Each side $= \frac{1}{\sqrt{2}}\{(h^2+k^2+l^2) + \sqrt{6(h^2k^2+h^2l^2+k^2l^2) - 3(h^4+k^4+l^4)}$

h, k, l being the three given straight lines.

[17] $r = \frac{16}{56}\sqrt{14}$; $R = \frac{45}{56}\sqrt{14}$. [18] 6. [19] $c = \frac{2(\text{area})}{a\sin B}$.

[20] $a = 2R\sin A$; $b = 2R\sin B$; $c = 2R\sin C$.

[28] $3:5:7$; $120°$. [29] $\sqrt{3}:\sqrt{2}$; $3^{\frac{1}{2}}:8$.

Ex. 15.

[30] $\frac{1}{8}(3+\sqrt{5})$; $\frac{1}{4}(2+\sqrt{2})$. [34] $\frac{9a^2}{4}(2-\sqrt{2})$.

[35] $2\left\{(a^2-c^2)^{\frac{1}{2}}+c\left(\pi-\cos^{-1}\frac{c}{a}\right)\right\}$.

Trigonometrical Tables.

Ex. 16.

[4] 20° 14′ 57″; 20° 10′ 38″·6. [5] 60° 5′ 12″; 13° 35′ 10″.
[9] 10° 9′ 9″; 82° 21′ 50″. [10] 84° 26′ 58″; 53° 2′ 10″.
[11] 41° 21′ 45″; 5° 25′ 2″.

Solution of Plane Triangles.

Ex. 17.

[1] B = 71° 46′. BC = 135·167. AC = 410·31.

[2] B = 47° 25′. AB = 437·958. AC = 322·466.

[3] A = 64° 40′ 47″. BC = 1115·45. AC = 527·754.

[4] A = 33° 6′ 25″. AB = 219·661. BC = 119·979.

[5] BC = 957·039. A = 88° 33′ 49″. B = 1° 26′ 11″.

[6] A = 5° 45′ 7″. B = 84° 14′ 53″. AC = 3544·06.

[7] A = 54° 5′ 1″. B = 35° 54′ 59″. AB = 721·1.

[8] C = 100° 22′ 45″. AB = 1758·78. AC = 1337·18.

[9] A = 127° 45′. AB = 115·905. BC = 407·401.

[10] C = 55° 55′ 43″. A = 35° 50′ 17″. BC = 372·5.

[11] B = 54° 10′ 56″, or 125° 49′ 4″. C = 73° 30′ 4″, or 1° 51′ 56″. AB = 393·755, or 13·3668.

[12] C = 27° 5′ 42″, or 152° 54′ 18″. B = 144° 34′ 43″, or 18° 46′ 7″. AC = 1590·1, or 882·72.

[13] B = 23° 8′ 33″. A = 32° 17′ 27″. BC = 171·268.

[14] C = 90°. B = 36° 46′. AC = 311·878.

[15] C = 76° 44′ 55″. B = 53° 59′ 5″. BC = 331·633.

[16] A = 157° 3′ 31″. C = 7° 43′ 15″. AC = 105·4075.

Ex. 17.

[17] $A=67° 58' 51''$. $B=59° 7' 4''$. $C=52° 54' 5''$. [18] $A=51° 8' 4''$. $B=68° 18' 50''$. $C=60° 33' 6''$.

[19] $C=158° 49' 55''\frac{1}{2}$; $B=11° 54' 29''\frac{1}{2}$.

Heights and Distances.

Ex. 18.

[1] 212·099 ft. [2] 732·21 ft.
[3] 75·6735; 43·4045; 61·988 yd. [4] 228·631 yd.
[5] 537·187 yd. [6] 49·643 ft. [7] 56·43 ft.
[8] 48° 22′. [9] 24109 × (Earth's rad.).
[10] 115·47; 9·503 yd. [11] 345·534 yd.
[12] 39·65 yd. [13] 910·82; 845·54 yd.
[14] 72 ft. [15] 101·891 ft. [16] 3513·325 yd.
[17] 48·633 ft. [18] 100·137 yd. [19] 441·955 yd.
[20] 160·85 ft. [21] 82·804 yd. [22] 578·6 yd.
[23] 420·7; 707·6 yd. [24] 6·143; 8·792 miles.
[25] 6·3397 miles. [26] N. 76° 56′ E.; 13·938 miles an hour.
[27] 2·36 miles. [28] 8448 miles.
[29] 1° 42′. [30] 7937·86 miles.
[31] SA = 1469·76; SB = 1650·10; SC = 1155·91 yd.
[32] SA = 1664·23; SB = 1493·45; SC = 2030·92 yd.
[33] SA = 896·34; SB = 1172; SC = 1506·34 yd.
[34] SA = 23·556; SB = 58·74; SC = 23·347 yd.
[35] 234·6 ft. [36] 200·21 ft.

Expansions, Series, &c.

Ex. 19.

[1] $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$;

$$\cos^8\theta = \frac{1}{128}\cos 8\theta + \frac{1}{16}\cos 6\theta + \frac{7}{32}\cos 4\theta + \frac{7}{16}\cos 2\theta + \frac{35}{128}.$$

[2] $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$;

$$\cos^7\theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta).$$

Ex. 19.

[3] $\sin^4\theta=\frac{1}{8}(\cos 4\theta-4\cos 2\theta+3)$;

$\sin^8\theta=\frac{1}{128}(\cos 8\theta-8\cos 6\theta+28\cos 4\theta-56\cos 2\theta+35)$.

[4] $\sin^6\theta=-\frac{1}{32}(\cos 6\theta-6\cos 4\theta+15\cos 2\theta-10)$;

$\sin^{10}\theta=-\frac{1}{512}(\cos 10\theta-10\cos 8\theta+45\cos 6\theta-120\cos 4\theta$
$+210\cos 2\theta-126)$.

[5] $\sin^5\theta=\frac{1}{16}(\sin 5\theta-5\sin 3\theta+10\sin\theta)$;

$\sin^9\theta=\frac{1}{256}(\sin 9\theta-9\sin 7\theta+36\sin 5\theta-84\sin 3\theta+126\sin\theta)$.

[6] $\sin^7\theta=-\frac{1}{64}(\sin 7\theta-7\sin 5\theta+21\sin 3\theta-35\sin\theta)$;

$\sin^{11}\theta=-\frac{1}{1024}(\sin 11\theta-11\sin 9\theta+55\sin 7\theta-165\sin 5\theta$
$+330\sin 3\theta-462\sin\theta)$.

[7] $\sin 4\theta=4\sin\theta\cos^3\theta-4\cos\theta\sin^3\theta$;
$\sin 9\theta=9\sin\theta\cos^8\theta-84\sin^3\theta\cos^6\theta+126\sin^5\theta\cos^4\theta$
$-36\sin^7\theta\cos^2\theta+\sin^9\theta$.

[8] $\cos 5\theta=\cos^5\theta-10\sin^2\theta\cos^3\theta+5\sin^4\theta\cos\theta$;
$\cos 6\theta=\cos^6\theta-15\sin^2\theta\cos^4\theta+15\sin^4\theta\cos^2\theta-\sin^6\theta$;
$\cos 7\theta=\cos^7\theta-21\sin^2\theta\cos^5\theta+35\sin^4\theta\cos^3\theta-7\sin^6\theta\cos\theta$.

[9] $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$;

$$\tan 8\theta=\frac{8\tan\theta-56\tan^3\theta+56\tan^5\theta-8\tan^7\theta}{1-28\tan^2\theta+70\tan^4\theta-28\tan^6\theta+\tan^8\theta}.$$

Ex. 20.

[1] $\dfrac{\sin\frac{nx}{2}\cos\frac{(n-1)x}{2}}{\sin\frac{x}{2}}$. [2] $\dfrac{x^{n+1}\cos(n-1)\theta-x^n\cos n\theta-x\cos\theta+1}{x^2-2x\cos\theta+1}$.

[3] $\dfrac{\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$. [4] 0. [5] $\cot\theta-2^n\cot(2^n\theta)$.

[6] $\cot\frac{1}{2}\theta-\cot(2^{n-1}\theta)$. [7] $\dfrac{1-\cos\theta\cos\phi}{1+\cos^2\theta-2\cos\theta\cos\phi}$.

Ex. 20.

[8] $\dfrac{\sin\frac{1}{2}(n+1)\phi\cos\left(\theta+\frac{1}{2}n\phi\right)}{\sin\frac{1}{2}\phi}$. [9] $\dfrac{n}{2}+\dfrac{\sin n\alpha\cos\{2\theta+(n-1)\alpha\}}{2\sin\alpha}$.

[10] $\dfrac{1}{2^{n-1}}\cot\dfrac{\theta}{2^{n-1}}-2\cot 2\theta$. [11] $e^{x\cos\theta}\cos(x\sin\theta)$.

[12] $\dfrac{\theta}{2}$. [13] $e^{x\cos\theta}\sin(x\sin\theta)$.

[14] $\tan^{-1}\left(\dfrac{x\sin\theta}{1+x\cos\theta}\right)$. [15] $\dfrac{\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}\left\{\cos\left(\dfrac{n+1}{2}\theta\right)+\sqrt{-1}\sin\left(\dfrac{n+1}{2}\theta\right)\right\}$.

[16] $\dfrac{\pi}{12}$. [17] $\dfrac{\pi^2}{6}$. [18] $\dfrac{\pi^2}{8}$.

[20] $x=n\sin a+\dfrac{n^2}{2}\sin 2a+\dfrac{n^3}{3}\sin 3a+\&c.$

[21] $\theta=m\cos\phi-\dfrac{m^2}{2}\sin 2\phi-\dfrac{m^3}{3}\cos 3\phi+\dfrac{m^4}{4}\sin 4\phi+\dfrac{m^5}{5}\cos 5\phi-\&c.$

Ex. 21.

[2] $\cdot 6931471$. [9] $a\,\mathrm{cosec}^2\dfrac{\pi}{2n}$. [10] na^{n-1}.

Ex. 22.

[1] $x=1\pm\sqrt{3};\ -2$. [2] $x=14\cos 20^\circ;\ -14\cos 40^\circ;\ -14\cos 18^\circ$.

[3] $x=2\cos 40^\circ;\quad 2\cos 80^\circ;\quad -2\cos 20^\circ$.

[4] $x=2;\quad -1;\quad -1$. [5] $x=\frac{1}{2}\{\pm\sqrt{2}\pm\sqrt{-2}\}$.

[6] $x=\cos 20^\circ\pm\sqrt{-1}\sin 20^\circ;\quad -\cos 40^\circ\pm\sqrt{-1}\sin 40^\circ;$
$-\cos 80^\circ\pm\sqrt{-1}\sin 80^\circ$.

Ex. 23.

[1] $x=a^2\sec^2\phi$; if $\tan\phi=\dfrac{b}{a}$.

[2] $x=2a^{\frac{1}{2}}\cos\frac{1}{2}\phi$; if $\sin\phi=\dfrac{b}{a}$.

[3] $\sin(x+\phi)=\dfrac{c}{a}\cos\phi$; if $\tan\phi=\dfrac{b}{a}$.

[4] $x=(a-b)\sec\phi$; if $\tan\phi=\dfrac{2(ab)^{\frac{1}{2}}}{a-b}\sin\frac{1}{2}C$.

[5] $\cos x=\dfrac{\cos A}{\sin\phi}\sin(B-\phi)$; if $\cot\phi=\cos c\tan A$.

APPLICATION OF ALGEBRA TO GEOMETRY.

[1] 7·4164, 4·5836 in. [2] 20, 21, 29.
[3] 21, 28, 35. [4] 9, 12.
[5] 8, 15. [6] 40, 42.
[7] 8·202, 3·048, 8·75. [8] 6, 8, 10.

[9] Let the hypothenuse $=c$, side of square $=s$; then the sides of the triangle are, (1) $\frac{c}{2}\left\{\frac{(c+s)^{\frac{1}{2}} \pm (c-3s)^{\frac{1}{2}}}{(c-s)^{\frac{1}{2}}}\right\}$;

(2) $\frac{1}{2}\left[s+(c^2+s^2)^{\frac{1}{2}} \pm \{c^2-2s^2-2s(c^2+s^2)^{\frac{1}{2}}\}^{\frac{1}{2}}\right]$.

[10] Sides $=\frac{1}{2}\left\{\frac{h}{2r}+r \pm \left(\frac{h^2}{4r^2}+r^2-3h\right)^{\frac{1}{2}}\right\}$.

[12] $3\frac{3}{7}$, $2\frac{4}{7}$. [13] $5·7\sqrt{2}$ or 8·06094.
[14] 12·579, 9·079. [15] 40·295, 9·705.

[16] 30, 40. [17] Base $=2d\left(1+\frac{p^2}{n^2-d^2}\right)^{\frac{1}{2}}$; sides $=d \pm n\left(1+\frac{p^2}{n^2-d^2}\right)^{\frac{1}{2}}$.

[18] $\frac{ap}{a+p}$. [19] Sides are, $a\left(\frac{\sqrt{5}+1}{2}\right)$, $a\left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{2}}$, a.

[20] If x, y, z be the sides, and h, k, l the perpendiculars on them;

$$x=\frac{2(hkl)^2}{hD}, \quad y=\frac{2(hkl)^2}{kD}, \quad z=\frac{2(hkl)^2}{lD};$$

where $D=\{hk+hl+kl)(hk+hl-kl)(hk-hl+kl)(hl-hk+kl)\}^{\frac{1}{2}}$.

[21] 13, 14, 15.

[23] Sides are $\left\{\frac{n^2a(ab-b^2+d^2)}{n^2(a-b)+b}\right\}^{\frac{1}{2}}$, $\left\{\frac{a(ab-b^2+d^2)}{n^2(a-b)+b}\right\}^{\frac{1}{2}}$; b, and $a-b$ being the segments of the base.

[24] Sides $=\left[2a^2-(m^4-4a^2p^2)^{\frac{1}{2}} \pm 2a\{a^2-p^2-(m^4-4a^2p^2)^{\frac{1}{2}}\}^{\frac{1}{2}}\right]^{\frac{1}{2}}$.

[25] If BP$=x$, CP$=a-x$, be the segments of the base BC, and $b>c$;

$$x=\frac{1}{4a}\{2a^2+c^2-b^2 \pm (4a^4+b^4+c^4-4a^2b^2-4a^2c^2-2b^2c^2)^{\frac{1}{2}}\}.$$

[26] The intercept is the hypothenuse of a right-angled triangle whose sides are 13·1355 and 9·1355.

[32] PMQ is the hypothenuse of a right-angled triangle, whose sides are $\frac{1}{2}\{b+2d \pm (b^2-4d^2)^{\frac{1}{2}}\}$; d being the distance of M from the line AA′ or BB′.

[34] If a be the radius of the quadrant—
side of square $=\frac{a}{2}\sqrt{2}$; radius of inscribed circle $=a(\sqrt{2}-1)$.

[35] Required radius $=\frac{ac}{a+c}$, if a be the radius and $2c$ the chord of sector.

[37] $\frac{1}{d}\{a(d^2-c^2)^{\frac{1}{2}}\pm c(d^2-a^2)^{\frac{1}{2}}\}$; a, c, d being the chds and diam^r.

[38] If $2a$, $2c$ be the chords, h the given distance;
diameter $=\{2(a^2+c^2+h^2)\}^{\frac{1}{2}}$.

[39] Radius $=\frac{2 \text{ area of the triangle}}{b+c-a}$.

[40] Chord $=\frac{1}{c}\{(c+r+r')(c+r-r')(c+r'-r)(r+r'-c)\}^{\frac{1}{2}}$.

[43] Side of square $=\frac{1}{\sqrt{2}}\{h^2+l^2\pm(4h^2k^2+4k^2l^2+2h^2l^2-h^4-l^4-4$

[45] Area of hexagon $=\frac{3\sqrt{3}}{2}$. [47] Radius $=\frac{\text{area of rhombus}}{\text{semi-perimeter}}$.

[51] Sum of circular areas $=\frac{\pi a^2}{4}\left(\frac{3+2\sqrt{2}}{4\sqrt{2}}\right)$, where $a=$ side of square.

[53] Volume of cone $=\frac{2\pi}{3}\left(\frac{R^5}{Rr-r^2}\right)$.

[55] The radii of the sections are, $\left(\frac{2a^3+b^3}{3}\right)^{\frac{1}{3}}$ and $\left(\frac{a^3+2b^3}{3}\right)^{\frac{1}{3}}$;
a, b being the radii of the ends of the frustum.

[56] Content of pyramid $=161{\cdot}589$.

ANALYTICAL GEOMETRY.

I. Straight Line.

Ex. 1.

[7] $y-6x+7=0$.

[8] If $y=ax+b$, $y=ax+c$; then $y=ax+\frac{1}{2}(b+c)$.

[9] $x_1=\frac{3}{7}$; $y_1=\frac{1}{7}$. [10] $11y-(8+5\sqrt{3})x=5(5-\sqrt{3})$.

[11] $y=\frac{a+b}{a-b}(x-c)$, if $c>a$. [12] $5y-4x=35$. [13] $\frac{10}{\sqrt{29}}$.

[14] $\frac{38}{85}\sqrt{5}$. [15] $\frac{8}{\sqrt{65}}$. [16] 90°. [17] $\frac{1}{18}$.

[18] $y-\frac{12}{13}=3{\cdot}566\left(x-\frac{30}{13}\right)$. [23] $\frac{3}{\sqrt{58}}$. [24] $\frac{8}{\sqrt{10}}$.

Ex. 1.

[25_1] Represents two straight lines, $x=\frac{9}{4}$; $y=-\frac{2}{3}$.

[25_2] Represents only the origin of coordinates.

[25_3] Represents the point of intersection of the two lines, $x=1$, $y=1$.

[25_4] Represents two straight lines whose equations are

$$y=(-2\pm\sqrt{3})x+(1\mp\sqrt{3}).$$

[25_5] Represents the point of intersection of the lines, $y=x+1$, $x=3$.

[25_6] Represents two straight lines, $y=x+2$; $y=x$.

[25_7] Represents the point of intersection of two lines, $x=1$, $y=\frac{1}{2}$.

[25_8] Represents three parallel straight lines, viz. the axis of x, and two lines on opposite sides of it, each at the distance $3c$.

[26] $90°$.

[27] $\tan\theta=\frac{m^2+1}{m^2-1}\tan\omega$; θ being the angle between the lines.

[28] $\frac{y-b}{x-a}=\frac{a}{b}$; $\quad p=\frac{ab\sin^2\omega}{(a^2+b^2+2ab\cos\omega)^{\frac{1}{2}}}$.

[31] $ae=bd$. [32] 10. [33] $\frac{2c^2}{21}$. [34] $\frac{1}{4}$. [35] $\frac{3}{2}$.

[36] $\frac{a^2}{2}\left\{\frac{\sin(\alpha-\alpha_1)\cos^2\alpha_2}{\sin(\alpha-\alpha_2)\sin(\alpha_1-\alpha_2)}\right\}$.

[37] Area $=\frac{1}{2}\left\{\frac{(b'-b)^2}{a'-a}+\frac{(b''-b')^2}{a''-a'}+\frac{(b-b'')^2}{a-a''}\right\}$. [38] $\frac{1}{2}ab$.

[39] $\frac{1}{2}\{(a'-a)(b'+b)+(a''-a')(b''+b')+(a-a'')(b+b'')\}$.

II. Circle.

Ex. 2.

[1] $h=-2$, $k=3$; $c=4$.

[2] $h=-6$, $k=4$; $c=2$.

[3] $h=1$, $k=-2$; $c=\frac{1}{2}$.

[4] $h=c$, $k=-3c$; radius $=c$.

[5] Coordinates of centre, each $=\frac{2}{3}a$; radius $=\frac{a}{\sqrt{3}}$; the coordinate axes are tangents to the circle.

[6] Coordinates of centre are, $\theta=\frac{1}{3}\pi$, $r=2$; radius of circle $=3$.

Ex. 2.

[7] $y=x-1$. [8] $x^2-9x+y^2-5y+14=0$.

[9] If a, b be the coordinates of the given point; the equation required is $x^2+y^2=ax+by$.

[10] If $(x-a)^2+(y-b)^2=r^2$, and $y=mx+c$, represent the circle and straight line respectively; then will

$$y-b=m(x-a)\pm r(1+m^2)^{\frac{1}{2}}.$$

[11] $4y=3x$. [12] $2x\pm 5^{\frac{1}{2}}y=9$.

[13] $x^2+y^2=\frac{2}{5}$. [14] $\frac{1}{c^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

[15] $\left(1-\frac{h^2}{c^2}\right)p^2-2kp+h^2+k^2=0$.

[16] If x_1y_1, x_2y_2 be the coordinates of the two given points, $ax+by=p$, the equation to the tangent;

then h, k, c are determined from the equations—

$$(x_1-h)^2+(y_1-k)^2=c^2,\ (x_2-h)^2+(y_2-k)^2=c^2,$$
$$(ah+bk-p)^2=(a^2+b^2)c^2.$$

[17] $x^2+y^2-2c(x+y)+c^2=0$. [18] $(x-\frac{1}{2}a)^2+y^2=c^2-\frac{1}{4}a^2$.

[19] $3y+5x+4=0$. [20] $2\left(c^2-\frac{a^2b^2}{a^2+b^2}\right)^{\frac{1}{2}}$.

[22] If α be the angular coordinate of the given extremity;

the polar equation is $r=\frac{c}{\sqrt{2}}\sec\left(\theta-\alpha\pm\frac{\pi}{4}\right)$.

[26] $x\cos(\alpha+\beta)+y\sin(\alpha+\beta)=c\cos(\alpha-\beta)$.

[27] Radius $=\dfrac{a\sin(\beta-\gamma)+b\sin(\gamma-\alpha)+c\sin(\alpha-\beta)}{4\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\alpha-\beta)}$.

[28] If h, k be the coordinates of the centre, ω the inclination of the coordinate axes, $c=$ radius of circle; the equation is

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega=c^2.$$

[29] $\omega=\frac{2\pi}{3}$; radius $=a$. [30] Diameter $=\frac{1}{\sin\omega}(b^2+c^2-2bc\cos\omega)^{\frac{1}{2}}$.

III. Parabola.

Ex. 3.

Let h, k denote the transformed coordinates of the vertex, L the latus rectum:

[1] $h=-\frac{1}{2}$, $k=0$; $L=\frac{2}{3}$. [2] $h=0$, $k=2$; $L=3$.

Ex. 3. Let h, k denote the transformed coordinates of the vertex, L the latus rectum:

[3] $h=\frac{3}{2}\sqrt{2}$, $k=-\sqrt{2}$; $L=2\sqrt{2}$.

[4] $h=\frac{\sqrt{2}}{4}$, $k=-\frac{3\sqrt{2}}{16}$; $L=\sqrt{2}$.

[5] $h=\frac{3}{4}\sqrt{2}$, $k=0$; $L=3\sqrt{2}$.

[6] $h=-\frac{3\sqrt{5}}{25}a$, $k=\frac{16\sqrt{5}}{75}a$; $L=\frac{3\sqrt{5}}{25}a$.

[7] $h=0$, $k=0$; $L=4c\sqrt{2}$.

[8] $y_1=a(\sqrt{5}-2)^{\frac{1}{2}}$, $y_2=a(\sqrt{5}+2)^{\frac{1}{2}}$.

[14] Ordinate = half the latus-rectum. [16] $c\sin\theta=a\cos^2\theta$.

[17] $x^2+y^2-(h+\frac{1}{2}l)x-\frac{1}{2}ky=0$; h, k being the coordinates of the point common to the three normals.

[19] $x^2+y^2-10ax+4ay-3a^2=0$.

[21] A straight line, $x=ac$.

[22] (1) A circle, $(x-a)^2+y^2=\frac{a^2}{m^2}$. (2) A parabola, $y^2=4ax+a^2n^2$.

[27] A parabola, $y^2=a(2x-a)$.

[28] A right line, $r\cos\theta=a+a'$, perpendicular to the axis.

[30] Distance $=4a$.

[32] $y=mx-(m^2+2)ma$; where $m=$ the tangent of the given $\angle$.

[33] A parabola, $y^2=a(x-a)$.

[37] $ay^2=\frac{4}{27}(x-2a)^3$ is the required equation; the locus of which is the evolute of the parabola.

[38] $2(a^{\frac{1}{2}}\sim m^{\frac{1}{2}})h^{\frac{3}{2}}$. [47] A parabola, $y^2=2a(x+a)$.

[49] $axy^2=(x^2+a^2)^2$, is the equation to the locus.

[50] $\frac{(x^2+y^2)^{\frac{1}{2}}}{x}=\frac{x^2+y^2+4a^2}{x^2+y^2-4a^2}$, S being the origin.

[53] $L=4r\sin^2\alpha$.

[65] $4a^2b^2\sin^2\omega\div(a^2+2ab\cos\omega+b^2)^{\frac{3}{2}}$. [66] $\frac{16\sqrt{2}}{3}a^2$.

[67] A parabola, $L=2\times$ rad. of sector, the centre of which coincides with the pole and focus.

[68] If $AB=c$, $BC=r$, $\angle ABC=\theta$; then $r=\frac{2c}{1+\cos\theta}$ is the equation to the locus.

IV. Ellipse.

Ex. 4. If h, k be the transformed coordinates of the centre of the ellipse, a, b the semi-axes:—

[1] $h=\frac{1}{3}$, $k=-\frac{1}{4}$; $a=\left(\frac{35}{72}\right)^{\frac{1}{2}}$, $b=\left(\frac{35}{48}\right)^{\frac{1}{2}}$.

[2] $h=1$, $k=-1$; $a=3$, $b=\frac{3}{\sqrt{2}}$.

[3] $h=\sqrt{2}$, $k=0$; $a=\sqrt{2}$, $b=\sqrt{3}$.

[4] $h=\sqrt{2}$, $k=2\sqrt{2}$; $a=\frac{1}{2}$, $b=\frac{1}{\sqrt{2}}$.

[5] $h=-\frac{\sqrt{2}}{3}$, $k=0$; $a=\frac{4}{3}\sqrt{2}$, $b=\frac{4}{3}\sqrt{6}$.

[6] $h=\left(1-\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}$, $k=-\left(1+\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}$; $a=(2-\sqrt{2})^{\frac{1}{2}}$, $b=(2+\sqrt{2})^{\frac{1}{2}}$.

[7] $h=\frac{3}{\sqrt{2}}$, $k=-\frac{1}{\sqrt{2}}$; $a=1$, $b=2$.

[8] $a=\frac{3}{2}$, $b=3$.

[10] Abscissa of $P=-14{\cdot}087$. [12] $\frac{1}{\sqrt{3}}$.

[20] $3y+x\sqrt{5}=9$. [27] $y=x+\sqrt{3}$.

[31] Area of triangle $=\frac{1}{2}(a^2\tan\theta+b^2\cot\theta)$.

[32] An ellipse, $y^2-b^2=c(x^2-a^2)$, c being negative.

[35] $y=2x-\frac{\sqrt{3}}{4}$.

[40] If ϕ be the inclination, $\cos\phi=m$, $\sin\phi=n$; the equation is $\frac{x}{m}-\frac{y}{n}=\frac{a^2-b^2}{(m^2a^2+n^2b^2)^{\frac{1}{2}}}$; the length $=\frac{b^2}{(m^2a^2+n^2b^2)^{\frac{1}{2}}}$.

[43] If h, k be the coordinates of the given point; the equation to the locus is $\frac{(2x-h)^2}{a^2}+\frac{(2y-k)^2}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$.

[44] The locus of Q is either of two circles, $x^2+y^2=(a\pm b)^2$.

[46] An ellipse, $(1+e)y^2+(1-e)x^2=a^2e^2(1-e)$.

[60] $\tan\frac{\gamma}{2}=\frac{5}{8}$. [63] $\frac{a^2}{x^2}+\frac{b^2}{y^2}=\left(\frac{a^2-b^2}{x^2+y^2}\right)^2$, is the equation to the locus.

[64] $9y+2x=0$. [69] $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

Ex. 4.

[72] $(a^2\sin^2\alpha+b^2\cos^2\alpha)(x^2+y^2)+2(a^2\sin^2\alpha-b^2\cos^2\alpha)xy=a^2b^2$.

[73] $\angle \text{ACP}=40^\circ\ 13'\ 34''$.

[80] An ellipse, major axis $=ae$, and similar to the original.

[83] Eccentricity $=\left(\dfrac{2e}{1+e}\right)^{\frac{1}{2}}$.

[86] $r+a=\dfrac{a(1-e^2)}{1+e\cos\theta}$. [89] $a=b\sqrt{2}$. [91] $\dfrac{2\text{CD}^2}{\text{AC}}$; $\dfrac{2\text{CD}^2}{\text{CP}}$.

[92] $t^2(x^2-a^2+y^2-b^2)^2=4(a^2y^2+b^2x^2-a^2b^2)$, if $t=\tan$ (given $\angle$).

V. Hyperbola.

Ex. 5. If h, k be the transformed coordinates of the centre of the hyperbola; a, b the semi-axes:—

[1] $h=0$, $k=\frac{1}{2}$; $a=b=\frac{1}{2}$.

[2] $h=0$, $k=0$; $a=b=2^{\frac{1}{4}}$.

[3] $a=b=\frac{1}{2}(242)^{\frac{1}{4}}$.

[4] $h=0$, $k=0$; a, $\frac{1}{2}a$ are the semiaxes.

[5] $h=0$, $k=0$; $a=(\sqrt{26}-5)^{\frac{1}{2}}\sqrt{14}$, $b=(\sqrt{26}+5)^{\frac{1}{2}}\sqrt{14}$.

[6] $h=0$, $k=0$; ratio of axes $=\tan\left(45^\circ-\dfrac{\alpha}{2}\right)$.

[7] $h=\dfrac{\sqrt{2}}{8}$, $k=0$; $a=\frac{3}{8}\sqrt{2}$, $b=\dfrac{\sqrt{3}}{4}$.

[8] $h=\frac{1}{4}(2-\sqrt{2})^{\frac{1}{2}}$, $k=\frac{1}{4}(2+\sqrt{2})^{\frac{1}{2}}$; $a=b=\dfrac{3\sqrt{2}}{8}$.

[9] $a=\sqrt{3}$, $b=\dfrac{1}{\sqrt{5}}$.

[12] $\perp^{\text{r}}=\left(\dfrac{m^2a^2-b^2}{1+m^2}\right)^{\frac{1}{2}}$; where $m=\tan$ (given inclination).

[17] Latus rectum $=8a$, eccentricity $=5^{\frac{1}{2}}$.

[18] If x' y' be the coordinates of the given point; those of the required point are $\dfrac{ay'}{b}$, $\dfrac{bx'}{a}$.

[21] Area $=\dfrac{2}{a}(a^2+b^2)(a^2+2b^2)^{\frac{1}{2}}$. [24] $\tan^{-1}\dfrac{\sqrt{5}}{2}$.

[32] The equation to the locus is $(x^2+y^2)^2=a^2x^2-b^2y^2$.

Ex. 5.

[33] The locus is a circle whose radius $=(a^2-b^2)^{\frac{1}{2}}$.

[34] A tangent at the vertex, $x=a$. [35] $13\cdot05$.

Sections of the Cone, etc.

Ex. 6.

[1] $60°$, $30°$; which correspond respectively to the elliptic and hyperbolic sections.

[4] $\cos^{-1}\frac{1}{3}$. [7] Latus rectum $=4d\sin^2\alpha$.

[8] $\theta=\cos^{-1}(\sqrt{2}\cos\alpha)-\alpha$.

[19] If l denote the length of the chord; then

$$l^2=\frac{4h(h^2+k^2)(m^2h+2mk^2+nhk^2)}{(nh^2-k^2)^2}.$$

[26] If the centre be the pole, the equation to the locus is

$$\frac{r}{c}=\frac{1-n^2\tan^2\theta}{1+n^2\tan^2\theta};\quad \text{where } n=\frac{a}{b},\ \text{and } c=\frac{a^2-b^2}{(a^2+b^2)^{\frac{1}{2}}}.$$

Ex. 7.

[1] $x^2+4y^2=2c^2$. [2] $4x^2y^2=2^{\frac{1}{2}}a(x^3-y^3)$.

[3] $\alpha=\tan^{-1}\frac{b}{a}$, $\beta=\pi-\alpha$; $a'=\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$.

Loci.

Ex. 8.

[1] If the distance between the two fixed points $=2c$, the given difference $=2a$, and the origin bisect the line $2c$; the equation to the locus is $(c^2-a^2)x^2-a^2y^2=(c^2-a^2)a^2$.

[2] If the middle point of the given side $(2a)$ be the origin, and the given sum $=\tan^{-1}m$; then $m(a^2-x^2-y^2)=2ay$.

[3] If the middle point of the given side $(2a)$ be the origin, and m the given difference; then $m(a^2-x^2)=2xy$.

[4] If the extremity of the base (c), at which is the less $\angle$, be the origin; then $y^2=3x^2-2cx$; a hyperbola.

[5] The locus is a circle; of which, rad. $=\frac{1}{3}\{3n^2-2(3a^2+h^2+k^2)\}^{\frac{1}{2}}$, and the coordinates of the centre are $x_1=\frac{1}{3}h$, $y_1=\frac{1}{3}k$; h, k being the coordinates of the vertex, and the middle point of the base $(2a)$ being the origin.

Ex. 8.

[6] A straight line perpendicular to, and bisecting, the base.

[7] If the fixed point be the origin of coordinates; $y=ax+b$ the equation to the given line; the equation to the locus is

$$y(1-ac)-x(a+c)=\frac{1}{2}b\operatorname{cosec}\frac{\pi}{10};\quad \text{where } c=\cot\frac{\pi}{10}.$$

[8] If the base $=2ae$, the sum of the 2 sides $=2a$; one extremity of the base being the pole, $r=\dfrac{2ae(1-e)\cos\theta}{1-e\cos 2\theta}$ expresses the locus.

[9] If the middle of the base $(2a)$ be the origin, and c the altitude; $c(a^2-x^2-y^2)=2y(a^2-x^2)$ expresses the locus.

[10] If the middle of the base $(2a)$ be the origin, x, y the coordinates of the vertex, $y'=mx'$ the equation to the given line; then

$$m(y^2-x^2+a^2)=(m^2-1)xy.$$

[13] If C be the origin, and $SH=2c$, the locus is expressed by the equation, $(x^2+y^2)^2=2c^2(x^2-y^2)$; the Lemniscate of Bernouilli.

[14] If the middle point of the chord $(2a)$ joining the two fixed points be the origin, α the angle at the centre subtended by this chord, β that by the given arc; the locus is a circle

$$(x^2+y^2-a^2)\tan(\alpha-\beta)=2ay.$$

[15] The curve is a circle.

[17] Radius of circle $=(a^2+b^2)^{\frac{1}{2}}$.

[20] If the middle point of $AB=2a$ be the origin, $\tan A=m$, $\tan B=n$, then $y^2=mn(x^2-a^2)$ is the equation to the locus.

[21] The locus of P is a hyperbola, whose axes coincide with those of the ellipse.

[22] If the given line be the axis of x, and a line perpendicular to it and passing through the given point the ordinate of which is c, be the axis of y, the locus is a parabola whose equation is

$$x^2+c^2=2cy.$$

[23] If A be the origin, $2a$ the diameter; $y^2(2a-x)=x^3$ is the equation.

[24] The vertex being the origin, and $4a$ the latus rectum of the parabola, then $y=\left(a-\dfrac{x}{4}\right)\left(\dfrac{x}{a}\right)^{\frac{1}{2}}$, is the equation to the locus.

[25] If A be the origin, $2a$ the diameter; $\dfrac{y}{2a}=\left(\dfrac{2a}{x}-1\right)^{\frac{1}{2}}$ is the equation.

[26] If C be the origin, $AC=a$, $CB=b$, XX' the axis of x; the locus of P is defined by the equation $xy=\pm(a+y)(b^2-y^2)^{\frac{1}{2}}$.

STATICS.

Forces acting in the same Plane.

Ex. 1.

[1] 4·63. [2] 90°. [3] $\sqrt{2}:1$. [4] 125° 41′. [5] 150°.

[6] 2·663; 47° 17′. [7] $\sqrt{6}:2:\sqrt{3}-1$. [8] 105°, 120°, 135°.

[9] 15·237 lb.; 28° 50′, ∠ between resultant and first force.

[10] 6·88; 102° 16′, ∠ between the resultant and first force.

[13] If α, θ be the ∠s between the directions of A, B; A, C respectively: $B=-A\cos\alpha \pm (C^2-A^2\sin^2\alpha)^{\frac{1}{2}}$; $\sin\theta=\frac{B}{C}\sin\alpha$.

[14] If there be n pairs of forces, the resultant acts in the direction of the perpendicular p, and is equal to $2np$.

[15] $p=\frac{1}{2}\left\{d+\left(P^2-d^2\sin^2\frac{\alpha}{2}\right)^{\frac{1}{2}}\sec\frac{\alpha}{2}\right\}$; $p'=\frac{1}{2}\left\{-d+\left(P^2-d^2\sin^2\frac{\alpha}{2}\right)^{\frac{1}{2}}\sec\frac{\alpha}{2}\right\}$

[16] 17·08276, 4·91724.

[17] $\cot(\alpha-2\theta)=3\cot\alpha$; α is the ∠ between the diameter AD and AB, θ between AC and AD.

[19] If $P=\frac{\mu}{CA}$, $Q=\frac{\mu}{CB}$; $R=\frac{\mu}{AB}\cdot\frac{4}{\sqrt{3}}$; ∠ between P, R=60°.

[20] R=5; ∠ between R and rod =83° 7′ 48″; portion of rod between R and the force (4) is 3·0217 feet.

Ex. 2.

[1] Force =50 lb., pressure =70·71 lb.

[2] 97° 11′, 124° 14′, 138° 35′.

[3] ∠ between the strings =60°.

[4] ∠ between CA, CB=120°, and CA=CB.

[5] If AB=$2a$, AC=c; $\frac{P}{W}=2\left(1-\frac{a^2}{c^2}\right)^{\frac{1}{2}}$.

[6] BC=·638 AB; ∠BAC=32° 32′ 3″; ∠ABC=24° 55′ 54″.

[7] If the inclination of AB to the horizon $=\alpha$; θ, ϕ the ∠s between AC, AP; and BC, BQ: $\cos\theta=\frac{W^2+P^2-Q^2}{2PW}$, $\cos\phi=\frac{W^2+Q^2-P^2}{2QW}$; hence ∠ACB$=\theta+\phi$, BAC$=90°-\theta\pm\alpha$, ABC$=90°-\phi\mp\alpha$, and AB$=a$, give AC, BC.

[8] Pressure = one of the weights. [9] 11 : 3.

Ex. 2.

[11] If BA, BC make $\angle$ s α, α' with the vertical upwards, and so of the rest; $P : Q : R = \cot\alpha + \cot\alpha' : \cot\beta + \cot\beta' : \cot\gamma + \cot\gamma'$.

[12] $2\tan^{-1}(\cdot 5)$, or $53^\circ\ 7'\ 48''$. [13] $W = 3\cdot 75$.

[14] $\cos\theta = \frac{P}{Q}\sin\alpha$; $R = P\cos\alpha - (Q^2 - P^2\sin^2\alpha)^{\frac{1}{2}}$; if R = pressure, $\theta = \angle BPE$, and α = the inclination of plane.

[15] $P : Q = \sin\beta : \sin\alpha$; tension $= P\sin\alpha$.

[16] Arc from A to $(2P) = 65^\circ\ 4'$. [17] $\tan PCA = \frac{P}{Q}$.

[19] $\frac{W\sin\beta}{\sin(\alpha+\beta)}$, on plane (α); $\frac{W\sin\alpha}{\sin(\alpha+\beta)}$, on (β).

Forces in one Plane, but not through one Point.

Ex. 3.

[1] The diagonal CA.

[3] Q acts in the direction DA, and Q : P = DA : CA.

[6] Tension $= \frac{W}{\sqrt{3}}$; on the two upper tacks, vert. press. $= \frac{1}{2}W$; on the two lower, $= 0$.

[7] If T be the tension of PA, T′ of QB; θ the inclination of PQ to the horizon; then, $2T\sin APQ = W\cos\theta = 2T'\sin BQP$.

[8] $\tan\theta = \frac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta}$; $R = \frac{W\sin\beta}{\sin(\alpha+\beta)}$, $R' = \frac{W\sin\alpha}{\sin(\alpha+\beta)}$; the lower end of beam resting on the plane (α).

[9] $\tan\theta = \frac{W'\cot\alpha - W\cot\alpha'}{W'+W}$; W, W′ being the weights of the spheres on the planes whose inclinations are α, α'.

[10] The centres C, A, B of the hollow and solid spheres are supposed to be in the same vertical plane; W, W′ the weights of the two latter; $CA = r$, $CB = r'$, $\angle ACB = \alpha$, $CAB = \beta$; φ = inclination of AC to the horizon; then

$$\tan\varphi = \frac{Wr}{W'r'}\operatorname{cosec}\alpha + \cot\alpha; \quad \text{and } \theta = \beta \sim \varphi.$$

[11] $AP = 1\cdot 838 \times$ radius.

[12] $CA = \frac{4}{\sqrt{3}}$, or $2\cdot 3094$ ft.; Tension $= \frac{5\sqrt{3}}{8}$(wt. of beam).

[13] Tension $= \frac{\sqrt{3}}{4}W$. [14] Tension $= \frac{36}{125}W$.

Ex. 3.

[15] $P=W\left(1-\frac{c}{a}\right)\sin\alpha$; where $a=$ length of beam, $c=$ the distance from the centre of gravity to its upper end.

[16] If $2a$, c be the lengths of beam and string; θ, ϕ their inclinations to the plane respectively; d the distance of the fixed point from the plane: the position is determined by
$2\sin(\phi-\theta)\sin\alpha=\cos\phi\cos(\theta+\alpha)$, and $2a\sin\theta+c\sin\phi=d$;
also, $\text{Tension}=\frac{\sin\alpha}{\cos\phi}W$; $\text{Pressure}=\frac{\cos(\alpha+\phi)}{\cos\phi}W$;
W being the weight of beam.

[17] $\cos^{-1}\left(\frac{7}{8}\right)^{\frac{1}{2}}$.

[18] If α, θ be the inclinations of DA, BA to the horizon, and W be the weight of beam; then, $2P\cos\frac{\alpha-\theta}{2}=W\cos\theta$.

[19] If the rod $BC=a$, A the fixed point, length of string $BAC=c$, θ the inclination of AB to the horizon; then, $AB=\frac{Qc}{P+Q}$, $AC=\frac{Pc}{P+Q}$; hence, the $\angle$ BAC may be found from the 3 sides of the ΔABC being known: $\theta=90^\circ-\frac{1}{2}BAC$.

[20] Let $AG=a$, G being the centre of gravity of rod whose weight is W; $c=$ a $\perp^r$ from D to the wall, θ the inclination of rod to the horizon, R, R′ the pressures at A and D; then

$$\cos\theta=\left(\frac{c}{a}\right)^{\frac{1}{3}};\quad R=W\left\{\left(\frac{a}{c}\right)^{\frac{2}{3}}-1\right\}^{\frac{1}{2}};\quad R'=W\left(\frac{a}{c}\right)^{\frac{1}{3}}.$$

[21] Let $\angle BAC=\alpha$, $ACE=\epsilon$, $W=$weight of beam;

$$\text{Tension}=W\frac{\cos\alpha}{2\sin(\alpha-\epsilon)}.$$

[22] $4\frac{1}{2}$ ft.

[23] $\tan\phi=2\tan\theta$; $2l\cos\theta=r\cos\phi+a$: where $2l=$ length of beam, $r=$ radius of hemisphere, and $a=$ a $\perp^r$ from its centre to the vertical plane.

[24] Let H, R, R′ be the horizontal force and pressures on sphere and plane respectively; $2a$ the length of beam, and W its weight, $\alpha=\angle BAC$, AC being horizontal: then

$$H=\frac{a\sin^2\alpha}{r}W,\ R=\frac{a\sin\alpha}{r}W,\ R'=\left(1-\frac{a}{r}\sin\alpha\cos\alpha\right)W.$$

[25] If $BC=c$, $BN=x$, the abscissa of W measured along BC, $r=$ radius; then $x=\frac{(W^2+Q^2)c^2-W^2r^2}{2W^2c}$.

Ex. 3.

[26] $7 \tan \theta = 9 \tan \theta'$, $9 \cos \theta + 7 \cos \theta' = 10$.

[27] $\tan^{-1}\left(\dfrac{ac+bc-a^2}{ac+bc-b^2}\right)$.

[28] If 2θ be the inclination, $2a$ the length of each beam, $r=$ radius of circle; $r \cos \theta = a \sin^3 \theta$.

[29] $\cdot 91 \times$ weight of 3 beams. [30] $\frac{1}{2} W \cot \alpha$.

[31] $\dfrac{a \sin 2\alpha \cos (\alpha+\beta)}{2b \sin \beta} \times$ (wt. of beam).

[32] $\cot \theta = \dfrac{3(4r^2h^2-a^4)^{\frac{1}{2}}}{4h^2-3a^2}$, θ being the inclination of triangle to the horizon.

[33] $\sin \theta = \left(\dfrac{2m}{a}\right)^{\frac{1}{2}} - 1$. [34] $17\cdot 79$ lb. nearly.

[35] If W, W′ be the weights of cylinder and beam, α, θ the inclinations of plane to horizon and of beam to the plane, a the distance of cent. of gravity of beam from the hinge; then,

$2 \cos (\theta+\alpha) \sin^2 \dfrac{\theta}{2} = \dfrac{Wr \sin \alpha}{W'a}$; r being the rad. of cylinder.

[36] $\tan \theta = \dfrac{Pr}{Wh}$; θ being the $\angle$ between the axis of bowl and the vertical, $r=$rad. and $h=$dist. of cent. of gravity of bowl from its geometrical centre.

[37] At distance from the ground $= \dfrac{a}{\sqrt{2}}$. [38] $\tan^{-1}\left(\dfrac{a^2+2ab}{b^2}\right)$.

[39] $60°$. [40] $\sin (\phi-\theta) = \frac{1}{6} \cos \theta = \sin \phi - \cos \phi$. [41] $30°$.

[42] $2a^3 : b(2a^2-c^2)$. [43] $a - b \tan \alpha : a + b \tan \alpha$.

[44] $53° 27' 23''$; $68° 5' 10''$ according as A is below or above C.

Ex. 4.

[1] Each distance $= 1$.

Centre of Gravity.

Ex. 5.

[3] In the $\perp^r$ from the right $\angle$ to the hypothenuse, at a dist. $2\frac{2}{3}$ from that point.

[4] $11\cdot 74$. [5] $1\cdot 3944$ ft. [6] $P : Q : R = \dfrac{\sin \beta}{p} : \dfrac{\sin \gamma}{q} : \dfrac{\sin \alpha}{r}$

Ex. 5.

[8] Equilateral. [9] Equilateral. [10] $\frac{1}{3}(h+h')$; $\frac{1}{3}(h-h')$.

[11] Distance from the middle point of base $=\frac{h}{3}\left(\frac{n^{\frac{1}{2}}+2}{n^{\frac{1}{2}}+1}\right)$ measured along h, the line bisecting the parallel sides.

[13] 18·02775 in.

[14] Altitude of $\Delta=(3-\sqrt{3})a=1{\cdot}268a$; $2a$ being a sideof square.

[15] In the diagonal drawn from A, the corner opposite to that cut off, and at a distance $\frac{19\sqrt{2}}{3}$ in. from A.

[16] Base of Δ cut off $={\cdot}634\times$ horizontal side of rectangle.

[17] $\frac{1}{3}(d-d')$; d, d', being the $\perp^{rs}$ from the opp. $\angle$s on the same diagonal.

[18] 5·059 in. [19] $\frac{(a^4+b^4+2a^2b^2\cos\alpha)^{\frac{1}{2}}}{2(a+b)}$.

[20] The point of support is in the line joining the weights 11, 23, and at a distance $=\frac{24a}{W+102}$ from the centre of the table, whose weight $=W$, and side $=a$.

[21] $\bar{x}=\frac{6}{5}a$, measured from the $\angle^r$ point without a weight, along a line drawn through the centre of the hexagon, whose side is a.

[23] A circle, whose radius $=\frac{1}{3}r$.

[24] It coincides with the centre of gravity of pyramid.

[25] $\frac{3}{4}W$, $\frac{1}{4}W$, at the base and apex. [27] $\frac{c}{4}\left(\frac{a^2+2ab+3b^2}{a^2+ab+b^2}\right)$.

[28] $\frac{1}{4}$ (sum of the altitudes). [29] $\frac{a^3+a^2b+ab^2+b^3}{a^2+ab+b^2}$.

[30] $\frac{3}{8}\left(\frac{4a^3+6a^2c+4ac^2+c^3}{3a^2+3ac+c^2}\right)$; $\frac{1}{2}a$. [31] $\bar{x}=\frac{1}{2}$ (radius).

[32] $\frac{9\sqrt{3}}{20}\times$ (edge of cube). [33] $\frac{185\sqrt{3}}{376}\times$ (edge of cube).

Ex. 6.

[1] $\bar{x}=\frac{4a}{3\pi}$; if $a=$rad. [2] $\bar{x}=\frac{4a}{3\pi}$; if $2a=$ major axis.

[3] $\bar{x}=\frac{3}{5}a$. [4] $\bar{x}=\frac{5}{7}h$, if $h=$ axis. [5] $\bar{x}=\frac{7}{6}a$.

Ex. 6.

[6] $\bar{x}=\frac{5}{6}a$. [7] $\bar{x}=\frac{c}{16}\left(\frac{34\log_e 2-15}{5\log_e 2-3}\right)$. [8] $\bar{x}=\frac{2}{3}\frac{\text{rad.}\times\text{chd}}{\text{arc}}$.

[9] $\bar{x}=\frac{\pi a}{4\sqrt{2}}$. [10] $\bar{x}=\frac{4a}{3\pi}$, $\bar{y}=\frac{4a}{3\pi}$.

[11] $\bar{x}=\frac{3}{5}a$, $\bar{y}=\frac{3}{4}a$; if $y^2=4ax$. [12] $\bar{x}=\frac{8m}{5a^2}$, $\bar{y}=\frac{2m}{a}$.

[13] $\bar{x}=\frac{4a'}{3\pi}$, $\bar{y}=\frac{4b'}{3\pi}$; $2a'$, $2b'$ being conjugate diameters.

[14] $\bar{x}=\frac{2}{3}\left(\frac{a}{\pi-2}\right)$, $\bar{y}=\frac{2}{3}\left(\frac{b}{\pi-2}\right)$; $2a$, $2b$ being the axes.

[16] If $\bar{x}$, $\bar{y}$ be measured along the axis and directrix respectively, $y=h$, $y=h'$ the equations to the two parallel lines;

$$\bar{x}=\frac{240a^4+40a^2(h^2-hh'+h'^2)+3(h^4-h^3h'+h^2h'^2-hh'^3+h'^4)}{480a^3+40a(h^2-hh'+h'^2)}$$

$$\bar{y}=\frac{3}{4}\left\{\frac{8a^2(h-h')+(h^2+h'^2)(h-h')}{12a^2+(h^2-hh'+h'^2)}\right\}.$$

[17] $\bar{x}=\frac{1}{2}\pi$, $\bar{y}=\frac{1}{8}\pi$. [18] $\bar{x}=\frac{3}{8}a$.

[19] $\bar{x}=\frac{3}{4}\frac{(a+h)^2}{2a+h}$; a being the rad. of sphere, and $a-h$ the height of segment.

[20] $\bar{x}=\frac{3}{8}a$; $2a$ being the major axis. [21] $\bar{x}=\frac{2}{3}a$.

[22] $\bar{x}=\frac{h}{3}\left(\frac{a^2+2b^2}{a^2+b^2}\right)$; a being greater than b, and $\bar{x}$ measured from the greater end.

[23] $\bar{x}=\frac{8ah+3h^2}{4(3a+h)}$; h being the length of the axis.

[24] $\bar{x}=\frac{3}{4}a\cos^2\frac{\alpha}{2}$; α being the $\angle$ between the bounding radii.

[25] $\bar{x}=\frac{5}{8}a$. [26] $\bar{x}=\bar{y}=\bar{z}=\frac{3}{8}a$.

[27] If r be the radius of the section of paraboloid made by the plane $x=c$; then $\bar{x}=\frac{2}{3}c$, $\bar{y}=\bar{z}=\frac{16r}{15\pi}$.

[28] $x=\frac{5}{4}a$, $\bar{y}=0$, $\bar{z}=\frac{5}{8}(c+m)a$.

[29] $\bar{x}=\frac{2a}{\pi}$. [30] $\frac{2\sqrt{2a}}{\pi}$.

[31] $\bar{x}=\frac{ac}{s}$, $\bar{y}=0$; the axis of x bisecting the arc, and $a=$radius.

Ex. 6.

[32] $\bar{x}=\frac{2}{3}a$, $\bar{y}=\left(\pi-\frac{4}{3}\right)a$. [33] $\bar{x}=\frac{m}{4}\left\{\frac{3\sqrt{2}-\log(\sqrt{2}+1)}{\sqrt{2}+\log(\sqrt{2}+1)}\right\}$.

[34] $\bar{y}=\frac{1}{s}\{\sqrt{2}+\log(\sqrt{2}+1)\}$; s being the length of the curve between the proposed limits.

[35] $\bar{x}=\frac{1}{2}c$. [36] $\bar{x}=\frac{2}{3}a$, from the vertex.

[37] $\bar{x}=\frac{1}{5}\left\{\frac{3a(a+m)^{\frac{3}{2}}}{(a+m)^{\frac{3}{2}}-m^{\frac{3}{2}}}-2m\right\}$. [38] $\bar{x}=\frac{2a}{15}\left(\frac{15\pi-8}{3\pi-4}\right)$.

[39] $\bar{x}=\frac{ab}{b-a}\log\frac{b}{a}$; a, b being the distances from the given point of the two extremities of the line, and the point being the origin.

[40] $\bar{x}=\bar{y}=\frac{3a}{2\pi}$. [41] $\bar{x}=\frac{5}{6}a$, a being the axis of cone.

[42] $\bar{x}=\frac{5a}{16}$, $\bar{y}=0$.

Ex. 7.

[1] The side (3) is horizontal. [2] $\cot\theta=\left(\frac{8a^2}{3b^2}-2\right)^{\frac{1}{2}}$. [3] $\frac{15}{8}$.

[4] Unstable. [5] Slant ht. of cone $=2$(rad. of hemisphere).

[6] Alt. of paraboloid $=\sqrt{1\cdot5}\times$ rad. of hemisphere. [7] $2:1$.

[8] If r, r' be the radii of the upper and lower hemispheres, the equilibrium is stable or unstable according as r is $<$ or $>\frac{3}{5}r'$.

[9] AD $=\sqrt{8\cdot5}\times$ AB. [10] Slant ht. of wall $=15\cdot45$ ft.

[11] $\cos^{-1}\left(\frac{1}{3}\right)$ or $70^\circ\ 32'$ nearly. [12] Roll. [13] $\frac{180^\circ}{n}$.

[14] $16^\circ\ 26'$. [15] $9m\cot^2\alpha$.

Ex. 8.

[1] $\frac{2bc}{a+b+c}\cos\frac{A}{2}$. [2] $c\frac{(a^4+b^4+2a^2b^2\cos C)^{\frac{1}{2}}}{ab+ac+bc}$.

Ex. 9.

[1] $3:1$. [2] $\frac{\pi a^2b^2}{3(a^2+b^2)^{\frac{1}{2}}}$.

[3] $\frac{\pi h}{3}(a^2+ab+b^2)$. [4] $\frac{4r}{3\pi}$; $\frac{2r}{\pi}$. [5] $4\pi r^2$.

Ex. 9.

[6] $\frac{1}{6}\pi r^3$. [7] $\bar{y}=\frac{3}{8}\times$ extreme ordinate.

[8] Surface $=8\pi a^2\left(\pi-\frac{4}{3}\right)$; volume $=\frac{1}{6}\pi a^3(9\pi^2-16)$.

[9] Surface $=\frac{32}{3}\pi a^2$; volume $=\frac{5}{2}\pi^2 a^3$.

[10] $\pi^2 abc$; a, b are the semiaxes, and c the dist. of the centre of ellipse from the axis.

Lever.

Ex. 10.

[1] 22 lb. [2] 19 : 24. [3] 8·928 ft. [4] $\frac{P}{Q}=1{\cdot}36$.

[5] $b-n+(b^2+n^2)^{\frac{1}{2}}$. [6] Where the wt. 7 acts. [7] $3\frac{10}{21}$ ft.

[8] At $\frac{2}{3}na$ from the end, at which the smallest wt. is placed.

[9] 20 lb. [10] 90 lb. [11] 20 lb.

[12] $2\frac{1}{20}$W. [14] $\sqrt{2}:\sqrt{3}$.

[15] Longer arm, 48° 22′ *below*; shorter, 18° 22′ *above* the horizon.

[16] 4·8496 lb.; 3·6372 lb. [17] $\sqrt{2}:1$; 120°. [18] 135°.

[22] 11 lb. 6 oz. [23] 6 lb.; 2 ft. 3 in., 1 ft. 6 in. [24] 2 : 1.

[25] If a be the length of the beam, the arms are $\frac{a\sqrt{p}}{\sqrt{p}+\sqrt{q}}$, $\frac{a\sqrt{q}}{\sqrt{p}+\sqrt{q}}$.

[26] $a\div(4n+1)^{\frac{1}{2}}$; where $2a$ is the length of the beam.

[27] Loss $=\frac{50}{m(m+1)}$ per cent. [29] 2, 8, 14, 20 &c. inches.

[30] $16\frac{8}{13}$, $15\frac{3}{7}$, $14\frac{2}{5}$, $13\frac{1}{2}$, $12\frac{12}{17}$, 12 inches. [31] $Q\sqrt{3}$.

Wheel and Axle.

Ex. 11.

[1] $1\frac{11}{37}$ in. [2] $28\frac{2}{3}$ lb. [3] $1234\frac{2}{7}$ lb.

[4] 264 lb. [5] 3 lb. [6] $\frac{1}{2}\left\{\left(\frac{\log 2p^8}{\log 2}\right)^{\frac{1}{2}}-1\right\}$.

[7] Angle θ between the greater wt. and vertical diameter of wheel is determined by $\tan\theta=\frac{3\sqrt{3}}{5}=\tan 46°\ 6'\ 7''{\cdot}6$.

Pulley.

Ex. 12.

[1] 120°. [2] 70·71 lb. [3] 100 lb. [4] 7.

Ex. 12.

[6] $64:63$. [7] 7. [8] $W=2^nP-\frac{1}{3}(4^n-1)w$.

[9] 756 lb. [10] 4 lb. [11] $1:247$.

[12] $W=(2^{n+1}-1)P+\{(n-1)2^n+1\}w$.

[13] $P:W=2:3^n-1$.

Inclined Plane.

Ex. 13.

[1] Height $=347$ ft., length $=485\cdot8$ ft. [2] $13\cdot4$ lb.

[3] $44°\ 25'$; $11°\ 32'$. [4] $20°\ 16'\ 54''$; $57°\ 12'\ 22''$.

[5] $\sin^{-1}\left(\frac{PR}{Wr}\right)$. [6] $P:W=\sqrt{2}:\sqrt{3}$.

Screw.

Ex. 14.

[1] $1:62\cdot83$. [2] $678\cdot58$ lb. [3] $3\cdot3157$ yd.

[4] $\cdot396$ lb. [5] $181\cdot865$ lb.

Friction.

Ex. 15.

[1] $W\left(\frac{\sin\alpha+\mu\cos\alpha}{\cos\beta+\mu\cos\beta}\right)$, $W\left(\frac{\sin\alpha-\mu\cos\alpha}{\cos\beta-\mu\sin\beta}\right)$.

[2] 25 lb. in a direction making 30° with the horizon. [3] $\frac{1}{\sqrt{3}}$.

[4] $\tan\theta=\frac{1-\mu\mu'}{2\mu'}$; press. on wall $=\frac{W\mu'}{1+\mu\mu'}$; press. on plane $=\frac{W}{1+\mu\mu'}$.

[5] To $\frac{5}{7}$ths of the ladder's length. [6] $\sin\theta=\frac{8\mu'(1+\mu)}{3(1+\mu\mu')}$.

[7] $W\left(\frac{\mu r}{l-\mu r}\right)$.

[8] If l, c be the length of the beam and string respectively, ϕ, θ their inclinations to the vertical, then $(4l^2-4c^2-\mu^2c^2)\tan^2\theta \mp 2\mu c^2\tan\theta+4l^2-c^2=0$; the $-$ or $+$ sign being taken according as the beam is on the point of sliding upwards or downwards respectively; also $l\sin\phi=c\sin\theta$.

[9] $\mu=\frac{1}{2}e^2$, e being the eccentricity. [10] $\mu=\frac{1}{2}\tan\frac{1}{2}\alpha$.

[11] $\frac{W'}{W}=\frac{1}{2}\left(\frac{\mu}{\mu'}-1\right)$; W being the weight of the upper cylinder, W′ the weight of each of the lower cylinders; μ, μ' the coefficients of friction respectively between the cylinders and between each cylinder and the plane.

DYNAMICS.

Collision or Impact of Bodies.

Ex. 1.

[1] $v=6\frac{17}{19}$ ft.; vel. lost by $A=1\frac{2}{19}$ft.; vel. gained by $B=1\frac{17}{19}$ft.
[2] $A:B=37:11$. [3] $38\frac{1}{4}$ ft. [4] $4\frac{1}{2}$ ft. [5] $\cdot08a$; $\cdot12a$.
[6] $\frac{35}{512}$ ft. [7] $29:36$. [8] $u=-\frac{1}{2}a$; $v=\frac{5}{2}a$. [9] $7:5$.
[10] 7. [11] $\cdot63$ lb. [12] $A=3B$, or $=\frac{1}{3}B$. [13] $B=2A$, or $\frac{3A}{2}$.
[14] $A:B=2n-1:1$. [15] $\cdot6$. [16] $u=-4\cdot\dot{4}\dot{5}$ ft.; $v=3\cdot9\dot{4}\dot{5}$ ft.
[17] $e=\frac{1}{6}=\frac{A}{B}$. [18] 123 ft. [19] $-\left(\frac{1+e}{1-e}\right)c$. [20] 30°.
[21] $a=\frac{1+e}{e}\cdot\frac{BV}{A+B}$; $b=\frac{(Be-A)V}{e(A+B)}$; where V is the given vel.
[22] $10\cdot4976:1$. [24] 30° and 45°. [25] $(1+e)\frac{AB}{A+B}$.
[26] $\frac{A+B}{2}\left[e\pm\left\{e^2-\frac{4AB}{(A+B)^2}\right\}^{\frac{1}{2}}\right]$; 1, $\frac{2(AB)^{\frac{1}{2}}}{A+B}$, are the limits.
[27] $v=\left(\frac{A}{B}\right)^{\frac{1}{2}}a$.
[28] $\tan\theta=\left(\frac{e^3}{1+e+e^2}\right)^{\frac{1}{2}}$; θ being measured from a diameter through the point of projection; $\frac{t_1}{t_3}=e$.
[29] $\cos\theta=\frac{2a^2b+(a-b)r^2}{2a\{ab(ab+r^2)\}^{\frac{1}{2}}}$; where $r=$ rad., a, b the distances of A, B from the centre, and θ the direction of projection.
[30] To be solved by a geometrical construction.
[31] $A:B:C:D=10:6:3:1$.
If m_r be the rth ball, &c.; $\frac{m_r}{m_{r+1}}=\frac{n-r+2}{n-r}$.
[32] $B=\frac{1}{e}A$; $C=\frac{1}{e^2}A$; $D=\frac{1}{e^3}A$; &c. Vel. required $=ae^{n-1}$.
[33] Time $=\frac{n(n-1)}{1.2}\frac{l}{a}$. [34] 90°.
[35] The direction of each ball after impact makes an $\angle 21°\ 3'\ 6''$ with the common tangent at the point of impact: vel. of $A=2a\times\cdot92796$; vel. of $(2A)=a\times\cdot92796$.

Ex. 1.

[36] $u^2=\frac{a^2}{4}\{1+e+(1-e)\cos\alpha\}^2+a^2\sin^2\alpha$; $v^2=\frac{a^2}{4}\{1-e+(1+e)\cos\alpha\}^2$

[37] $u=a\left\{\sin^2\alpha+\left(\frac{A\cos\alpha}{A+B}\right)^2\right\}^{\frac{1}{2}}$.

[41] If c be the given distance; A will overtake B at a distance $=c\frac{2C(A-B)}{3AC-B(A+B+C)}$.

[44] $mu^2+m'v^2=ma^2+m'b^2-\frac{(1-e^2)mm'(a-b)^2}{m+m'}$.

Uniformly Accelerated Motion and Gravity.

Ex. 2.

[1] $s=1948\cdot1$ ft.; $v=354\cdot2$ ft. [2] $t=15\frac{1}{2}$ sec.; $s=3883$ ft.

[3] Vel. $=170\cdot2$ ft.; mom. $=10$ tons $12\frac{3}{4}$ cwt. [4] $13:29$.

[5] $275\cdot4$ ft. [6] $225\cdot4$ ft. [7] 15970 ft.

[8] Alt. $=477\cdot9$ ft.; $t=5\cdot449$ sec. [9] $t=1\cdot778$ sec.

[10] $\frac{g}{g'}=\left\{\left(1+\frac{mT}{4a}\right)\pm\frac{mT}{4a}\left(1+\frac{8a}{mT}\right)^{\frac{1}{2}}\right\}^2$. [11] $63\cdot6$ ft.

[12] $V=28\cdot4$ ft. [13] Ht. $=100\cdot59$ ft.; $t=2\cdot408$ sec.

[14] $V=\left\{2g\left(a+\frac{1}{2}gt^2\right)\right\}^{\frac{1}{2}}$; $T=2\left\{\frac{2}{g}\left(a+\frac{1}{2}gt^2\right)\right\}^{\frac{1}{2}}$: where a is the distance of the given point from the point of projection, and $2t$ the given time.

[15] $120\cdot4$ ft. from the bottom of steeple. [16] $1\cdot42$ sec.

[17] $321\cdot9$ ft. [18] $f=\frac{P-Q}{p-q}$; $V=\frac{Q(2p-1)-P(2q-1)}{2(p-q)}$.

[19] $72\cdot25$ ft.

[20] At the distance $\frac{h}{2}\cdot\frac{2a^2+2ac+gh}{(a+c)^2}$, from the upper extremity.

[21] Vel. $=\frac{a}{n}+(2ga)^{\frac{1}{2}}$. [22] $V=100\cdot216$ ft. [23] $1173\cdot8$ ft.

[24] If c be the given vel.; $e=\frac{(gt+c)^2-c^2}{(gt+c)^2+c^2}$; dist. of plane $=\left(\frac{e^2}{1-e^2}\right)\frac{c^2}{2g}$.

[25] $e=\sqrt{\cdot4}$; $V=(10g)^{\frac{1}{2}}=17\cdot94$ ft. [26] $\frac{49\cdot7}{1-e^2}$ ft. [27] $106\frac{1}{4}$ ft.

[28] Let a, b, x be the distances from the horizontal plane of the given points and point of meeting respectively; then

$$x=\frac{10ab-a^2-9b^2}{16b}.$$

Ex. 2.

[29] $2(ab)^{\frac{1}{2}}$; a, b being the altitudes of the given points.

[30] If O be the point of meeting, $PO = \frac{m'^2}{m'^2 - m^2}(2a - h)$; where m, m' are the masses of the balls, $PQ = 2a$, and h = ht. due to the given velocity.

Ex. 3.

[1] $33^\circ\ 59'\ 56''\cdot 4$. [2] $s = 123\cdot 227$ ft.; $t = 5\cdot 46$ sec.; $v = 118\cdot 67$ ft.

[3] $7306\cdot 2$ ft. [4] 32 ft. [5] $\sin^{-1}\left(\frac{1}{3}\right)$, or $19^\circ\ 28'\ 16''$.

[6] The upper extremity of the part is $2\frac{1}{2}$ ft. from the top of the plane.

[7] $\frac{1}{9}l$, $\frac{3}{9}l$, $\frac{5}{9}l$.

[9] Distance from the bottom of plane $= \frac{P+W}{P}\left(\frac{hl}{h+l}\right)$;

$$\text{Time} = \frac{(P+W)l}{P}\left\{\frac{2Ph}{(Pl - Wh)(h+l)g}\right\}^{\frac{1}{2}}.$$

[10] At $\left(1 - \frac{1}{4n}\right)l$, distance from the bottom.

[11] At a Distance $= c\left(\frac{l - an - \frac{1}{2}n^2 g \sin\alpha}{a + c + ng\sin\alpha}\right) - \frac{1}{2}g\sin\alpha\left(\frac{l - an - \frac{1}{2}n^2 g\sin\alpha}{a + c + ng\sin\alpha}\right)^2$ from the lower extremity.

[12] $l = \frac{V^2}{2g\sin\alpha}\cdot\frac{\left(\frac{4}{9} - e^2\right)}{(1 - e^2)}$. [13] Ht. $= a$.

[14] If C be the rt. $\angle$, CA vertical; $\angle ACP = 45^\circ - \frac{1}{2}A$, P being the required point.

[15] $\frac{\text{vertical side}}{\text{horizontal side}} = \frac{3}{4}$.

[16] $x = \frac{c\sin\beta}{\sin\alpha\cos^2(\alpha+\beta)}$; α, β being the inclinations of $CA = x$, $CB = c$, to the horizon.

[17] $\frac{a}{16}$; time $= \left(\frac{5}{2}a + 2d\right) \div (ga\sqrt{3})^{\frac{1}{2}}$.

[19] $\angle$ between the chord and diameter $= 60^\circ$.

Ex. 3.

[20] Diameter's inclination to the vertical $=85^\circ\ 4'\ 43''\cdot6$.

[22] $A:B=\sqrt{3}-1:2=\cdot366:1$.

[25] If θ be the inclination of the required diameters to the axis of x, and n sec. the given time;

$\cos\theta=\pm\left[\frac{e^2+1}{2e^2}\pm\left\{\left(\frac{e^2+1}{2e^2}\right)^2-\frac{g^2n^4+16b^2}{g^2n^4e^2}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}$; and

$\cos\theta=\pm\left(\frac{e^2+1}{2e^2}\right)^{\frac{1}{2}}$, when the time is a minimum.

[26] $t=\left\{\frac{a(\pi^2+4)}{g\sqrt{3}}\right\}^{\frac{1}{2}}$.

[27] The locus is an ellipse or hyperbola, according as the given constant c is greater or less than $\frac{4a}{g}$, $2a$ being $=AB$; the equation is $y^2=\left(\frac{gc}{4a}-1\right)(a^2-x^2)$, the origin bisects AB and y is horizontal.

[28] $x^2-y^2=a^2$; $2a=$ distance between the two given points, and x is measured from the middle point between them.

[29] $V_1=11\cdot852$ft.; $V_2=22\cdot304$ft.; $t_1=2\cdot025$sec.; $t_2=2\cdot545$sec.

[30] $\frac{W}{4\cdot65}$.

[31] $9\cdot12904$ lb.; $8\cdot87096$ lb.

[32] $s=203\cdot68$ ft.; $v=46\cdot92$ ft.

[33] P will then descend through a space $=\frac{(P-Q)(Q+P-p)}{(P+Q)(Q-P+p)}a$; after which, it will ascend.

[34] Ht. $=\frac{Q+P}{Q-P}a$; $t=\frac{Q+P}{Q-P}\left(\frac{2a}{g}\right)^{\frac{1}{2}}$.

[35] Draw a line from the upper extremity of the vert. diam. to the given point, and produce it to meet the circumference; the produced part is the line of quickest descent.

[36] The axis.

[37] $\theta=\cos^{-1}\left(\frac{1}{2e}\right)$.

SIMPLE PENDULUM.

Ex. 4.

[1] $\frac{1}{4}\left\{\frac{3g}{2}(9a+6\sqrt{3}\,b+8c)\right\}^{\frac{1}{2}}$; where a, b, c are the lengths of the planes A, B, C.

Ex. 4.

[2] $e=\frac{2b}{a}-1$. [4] 2·4462 in.; 1739·52 in. [5] 4·289 sec.

[6] 52·036 in. [7] $l=\frac{18}{\pi^2}=1·82378$ ft. [9] 568·08 sec.

[10] Diminished by ·018 in. [11] $3\frac{1}{3}$ miles. [12] 13·24 miles.

[13] $2\frac{1}{4}$ miles, nearly. [14] 23 hr. 56 m. 4 sec.; ·2133 in.

[15] 144 : $145\frac{1}{176}$. [16] $l=\left(\frac{n}{n+2}\right)^2 L$. [17] 28·139 in.

[18] $\frac{g}{g'}=1+\frac{2n}{86400}$, nearly. [19] $\frac{720}{mn}$ in. [20] 3986·4 ft.

[21] $2:\pi$. [22] ·94 of a mile.

[23] 1·95 miles *below*, or ·98 mile *above* the earth's surface.

[25] Earth's radius $=\frac{ht_2^2}{t_2^2-t_1^2}$; where t_1 is the time of an oscillation at the surface, and t_2 at the given depth h.

[26] $\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$; axis of cycloid being $2a$.

Projectiles.

Ex. 5.

[1] R = 55·916 ft.; T = ·965 sec; H = 3·745 ft.

[2] V = 126·86 ft.; H = 125 ft.; T = 5·573 sec.

[3] $\alpha=74°\ 33'\ 48''$; V = 250·46 ft.; H = 905·36 ft.

[4] 841 ft. [5] 60°.

[6] $\tan^{-1}\frac{4}{3}$, or 53° 8′; $\tan^{-1}4$, or 75° 58′.

[7] V = 645·1 ft.; $\alpha=8°\ 5'\ 4''$.

[8] V = 165·48 ft.; T = 9·778 sec.

[9] 179·62 ft. [10] $2450g\tan\alpha$.

[11] Lat. rect. = 11222 ft. nearly; the coordinates of focus are, $x_1=9718·9$ ft.; $y_1=5611·2$ ft.

[12] If x_1y_1, x_2y_2 be the coordinates of the two given points;

$$\tan\alpha=\frac{x_1^2y_2-x_2^2y_1}{x_1x_2(x_1-x_2)};\qquad V^2=\frac{g}{2}\cdot\frac{x_1x_2(x_1-x_2)}{x_1y_2-x_2y_1}\sec^2\alpha.$$

[13] $\tan^{-1}2$, or 63° 26′ 6″. [14] $1:4\cos^2\alpha$. [15] 60°.

[16] $\tan^{-1}\sqrt{2}$, or 54° 44′; $90°-\tan^{-1}\sqrt{2}$, or 35° 16′.

Ex. 5.

[17] $56° 19'$. [18] $\frac{ab}{a+b}$.

[19] The direction, $\theta = 157°30'36''$; vel. $= 81\cdot177$ ft.; ht. $= 247\cdot13$ ft.

[20] $30°$; $60°$. [21] $\frac{t}{t'} = \frac{\sin(\alpha-\beta)}{\cos\alpha}$. [23] Range $= \frac{2h\sin 2\alpha}{1-\epsilon}$.

[25] Ht. $= \frac{n^2 g}{2} - \frac{a}{\sqrt{3}}$. [26] Vel. $= \left(\frac{ag}{8}\right)^{\frac{1}{2}}$; a being the ht. of tower.

[27] $R = 128$ ft.; time $= 5\cdot12$ sec.

[28] At dist. $= \frac{8V^2}{3g}\sin(30°+\alpha)\cos\alpha$.

[29] Ht. of mountain $= 3600$ ft.; greatest ht. of projectile $= 846\cdot86$ ft.

[30] $V = 124\cdot68$ ft.; $t = 5\cdot4778$ sec. [31] $503\cdot2$ ft.

[32] $21\cdot637$ ft. from the base of house; $t = 1\cdot1$ sec.

[33] $AD = 1\cdot7071\ AC$; $t = \cdot487 AC^{\frac{1}{2}}$ sec.

[34] If $\theta =$ inclination of the plane, a the given base, and u the vel. of projection; $\tan 2\theta = \frac{u^2}{2ga}$.

[35] The value of $(x^2+y^2)^{\frac{1}{2}}$, deduced from, $y = x - \left(\frac{x}{160}\right)^2 g$; and $y = \left(\frac{5}{\sqrt{2}} - \frac{x}{160}\right)^2 g$.

[36] Ht. $= 4h\frac{\cos\alpha\cos\beta\cot(\alpha+\beta)}{\sin(\alpha+\beta)}$; α, β being the $\angle$s of projection.

[37] $\bar{y} = \frac{2Au+Bv}{\sqrt{3}Bv}\bar{x} - \frac{2(A+B)^2}{3B^2v^2}g\bar{x}^2$; $\bar{x}$, $\bar{y}$ being the coord. of the centre of gravity.

[38] Ht. $= 1917\cdot088$ ft.; $H = 7337\cdot7$ ft.

[39] If u, v be the vels. of projection of A, B, respectively; $\sin\alpha = \frac{u}{v}$.

[40] If the point of projection be the origin, t the given time; the locus is the circle, $x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = V^2t^2$.

[41] An ellipse, $x^2 = 4(hy - y^2)$.

[42] A parabola, $r = \frac{2h}{1+\cos\theta}$; $\theta = \frac{\pi}{2} - \iota$.

[43] $R = \left(\frac{4}{3}\right)^{\frac{3}{2}}\epsilon l$; where $l =$ length of plane, $\epsilon =$ elasticity.

[44] Dist. $= 4h(1+\epsilon)\epsilon\sin\theta$; where $h =$ given ht., and $\theta =$ inclination of plane.

[45] Time $= \left(\frac{8h}{g}\right)^{\frac{1}{2}}$; dist. $= 4h$. [46] $\tan\alpha = \frac{1}{2}(3\tan\iota + \cot\iota)$.

[47] $\epsilon = \cdot5$; one ball is double of the other.

Ex. 5.

[49] $T=\frac{2}{g}\left\{\frac{uv\sin(\alpha\backsim\beta)}{u\cos\alpha+v\cos\beta}\right\}$.

[50] At a dist. from the point of projection $=(1-e)h\sin 2\alpha$;
$t=\left(\frac{8h\sin^2\alpha}{g}\right)^{\frac{1}{2}}$.

[52] 2·1181 miles. [53] 1·554 lb. [54] 4lb. 3oz.; $t=12\frac{2}{3}$sec.

ROTATION OF BODIES.

I. Moment of Inertia.

Ex. 6.

[1] $k^2=\frac{1}{3}a^2$; a being the length of the rod.

[2] $k^2=\frac{1}{2}a^2$; a being the radius of circle.

[3] $k^2=\frac{1}{2}a^2-\frac{c(a^2-c^2)^{\frac{1}{2}}}{2\sin^{-1}\frac{c}{a}}$; a being $=$ radius, and $2c=$ chord of arc.

[4] $k^2=\frac{r^2}{s^2}(s^2-c^2)$; r, c, s being the radius, chord and arc.

[5] $k^2=2r^2\left(1-\frac{c}{s}\right)$; r, c, s as in the last problem.

[6] $k^2=\frac{1}{4}a^2+c^2$; a being the radius of circle.

[7] $k^2=\frac{1}{4}b^2$; $k'^2=\frac{1}{4}a^2$, about the major and minor axes respectively.

[8] $k^2=\frac{1}{4}(a^2+b^2)$; $2a$, $2b$ being the axes of ellipse.

[9] $k^2=\frac{a^2}{24}$; a being the base of triangle.

[10] $k^2=\frac{1}{12}(3a^2+3b^2-c^2)$; c being the side opposite to the angle at which is the axis.

[11] $k^2=\frac{1}{36}(a^2+b^2+c^2)$.

[12] $k^2=\frac{1}{3}(a^2+b^2)$; $2a$, $2b$ being adjacent sides.

[13] $k^2=\frac{a^2}{12}\left(\frac{2+\cos\frac{2\pi}{n}}{1-\cos\frac{2\pi}{n}}\right)$; a being the length of each side.

Ex. 6.

[14] $k^2=\frac{1}{2}(a^2+b^2)$; a, b being the radii of the annulus.

[15] $k^2=\frac{3}{7}a^2+\frac{1}{5}b^2$; a, b being the extreme ordinates.

[16] $k^2=\frac{2}{5}a^2$; a being the radius of sphere.

[17] $k^2=\frac{2}{5}\left(\frac{a^5-b^5}{a^3-b^3}\right)$; a, b being the radii of shell.

[18] $k^2=\frac{1}{2}a^2$; a being the radius of cylinder.

[19] $k^2=\frac{1}{4}a^2+\frac{1}{3}c^2$; a being the radius, and $2c$ the length of cylinder.

[20] $k^2=\frac{3}{10}a^2$; a being the radius of base.

[21] $k^2=\frac{3}{20}(a^2+4c^2)$; a = radius of base, and c = altitude.

[22] $k^2=\frac{1}{3}b^2$; b being the rad. of the base of paraboloid.

[23] $k^2=\frac{1}{2}(n+1)(n+2)k'^2$; $n=7$, or -10.

II. Centre of Oscillation.

L is used to express the length of the seconds' pendulum.

Ex. 7.

[1] $t=\left\{\frac{3}{4L}\left(d+\frac{a^2}{12d}\right)\right\}^{\frac{1}{2}}$; d being the altitude and a the base of triangle.

[2] $t=\left(\frac{5\sqrt{3}}{12}.\frac{a}{L}\right)^{\frac{1}{2}}$; a being a side of the triangle.

[3] $t=\left(\frac{17c}{12L}\right)^{\frac{1}{2}}$; c being a side of the hexagon.

[4] $t=\left(\frac{2r}{L}\right)^{\frac{1}{2}}$; r being the radius of the arc.

[5] $t=\left(\frac{2\sqrt{2a}}{3L}\right)^{\frac{1}{2}}$; a being a side of the cube.

[6] $t=\left(\frac{12h^2+a^2}{15hL}\right)^{\frac{1}{2}}$; h being the altitude, and a a side of base.

[7] $t=\left(\frac{3a}{2L}\right)^{\frac{1}{2}}$; a being the radius of cylinder.

Ex. 7.

[8] $t=\left(\frac{7a}{5L}\right)^{\frac{1}{2}}$; a being the radius of sphere.

[9] $t=\left\{\frac{23a^2+2c^2}{5(16a^2+c^2)^{\frac{1}{2}}L}\right\}^{\frac{1}{2}}$; a = rad. of base, c = alt. of cone.

[10] $t=\left\{\frac{b^2+3c^2}{4cL}\right\}^{\frac{1}{2}}$; b = rad. of base, c = axis of paraboloid.

[11] $t=\left(\frac{152a}{225L}\right)^{\frac{1}{2}}$; a being the radius of sector.

[12] $t=\left(\frac{5a}{6L}\right)^{\frac{1}{2}}$; a being the length of the rod.

[13] $42\frac{3}{8}$ in. nearly.

[14] $t=\sqrt{7}$ sec.

[15] 1·0589 sec.

[16] 1·2198 sec.

[17] $x=\frac{l}{n}\{(m^2+mn)^{\frac{1}{2}}-m\}$; m = mass of bob, n = mass of given wt.; l, x the distances of m, n from the point of suspension.

[18] $l=\frac{5}{4}\left(\frac{a^3+a^2b+ab^2+b^3}{a^2+ab+b^2}\right)$; a, b being the radii of the circles.

[19] $\frac{t}{t'}=\left(\frac{5}{6}\right)^{\frac{1}{2}}$.

[20] Base of $\Delta = 2\,(\text{Area})^{\frac{1}{2}}$.

[21] $2\sin^{-1}\frac{3}{4}$.

[23] Diam. $= 2 \times$ altitude.

[24] Alt. $h=\left\{\frac{5}{4}\left(60^2L-\frac{3V}{5\pi}\right)\right\}^{\frac{1}{3}}$; V being the volume.

[25] The distance x from the upper extremity of the cylinder is given by the equation,

$$x^2-\left\{2c-n^2\left(\frac{a^2}{4c}+\frac{4c}{3}\right)\right\}x=(n^2-1)\left(\frac{1}{4}a^2+\frac{4}{3}c^2\right);$$

where a = rad. of cylinder, and $2c$ = its length.

[26] If OA $=a$, OA$'=x$; m, m' the masses of the spheres A, A$'$;

$$x=\frac{1}{m'}\left\{m(m+m')a^2+\frac{2}{5}m'(mr^2+m'r'^2)\right\}^{\frac{1}{2}}-\frac{m}{m'}a.$$

[27] $n=2$.

[28] $l=\frac{2}{3}\frac{a^3+b^3}{(a^4+b^4+2a^2b^2\cos\theta)^{\frac{1}{2}}}$.

[29] $\frac{1}{5}\times$ length of rod.

III. D'Alembert's Principle.

Ex. 8.

[1] The rod will oscillate isochronously with a simple pendulum whose length $=\frac{P\times AP^2+P'\times AP'^2}{P\times AP+P'\times AP'}$.

Ex. 8.

[2] If m, m' be the masses of the 2 particles,
. α, α' . . inclinations of the planes to the horizon,
. x, x' . . distances of the particles at the time t from the common summit of the planes; then

$$(m+m')\frac{d^2x}{dt^2}=g(m\sin\alpha-m'\sin\alpha'); \text{ Tension}=\frac{mm'}{m+m'}(\sin\alpha+\sin\alpha')g.$$

[3] $\theta=\frac{1}{2}gt^2\left(\frac{ma-m'a'}{ma^2+m'a'^2+Mk^2}\right)$, the whole $\angle$ through which the wheel and axle have revolved in the time t;

$$\frac{T}{mg}=\frac{m'a'(a+a')+Mk^2}{ma^2+m'a'^2+Mk^2}; \qquad \frac{T'}{m'g}=\frac{ma(a+a')+Mk^2}{ma^2+m'a'^2+Mk^2}.$$

[4] Accelerating force $=\frac{(m-m')ga^2}{(m+m')a^2+Mk^2}$; A being the fixed pulley, a its radius, and Mk^2 its moment of inertia;

Tension of AP $=\frac{(2m'a^2+Mk^2)mg}{(m+m')a^2+Mk^2}$; of BQ $=\frac{(2ma^2+Mk^2)m'g}{(m+m')a^2+Mk^2}$.

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[5] The point required is at a distance $=\frac{2}{3}a$ from the middle point of the fixed side; a being a side of square.

[6] Space $=92\cdot856$ ft.

[7] Weight of cylinder $=37\cdot258$ lb.

[8] Time $=33\cdot488$ sec.; velocity $=2\cdot986$ ft.

[9] Velocity $=12\cdot29$ ft.; no. of revolutions in 1 m. $=122\cdot022$ nearly.

[10] 2·79 sec. [11] Weight $=84\cdot109$ lb. nearly.

[12] Radius of wheel $=16\cdot4$ ft.; its wt. $=5\frac{1}{3}$ lb.; pressure $=610\cdot4$ lb.

[13] 109 : 64.

HYDROSTATICS.

Pressure on Surfaces.

Ex. 1.

[1] 1 : 2 : 3; the second term corresponding to the vertical side.

[2] 2 : 1. [3] The line bisects the axis. [4] $\frac{1}{3}A(h+k+l)\rho$.

[5] 1 : $\sqrt{2}$. [6] 80·12 in. [7] Base of triangle $=5\frac{1}{16}$.

[8] The breadths are 8·9443, 3·7048, 2·8428, 2·3966, 2·1115.

[9] L : M : N = 1 : 3 : 5. [10] Radius $=4\cdot9324$.

Ex. 1.

[11] Press. on large sq. : press. on ⊙ : press. on small sq. $=4:\pi:2$.

[12] $4:5$. [13] Depth of line $=\left(\frac{m}{m+n}\right)^{\frac{2}{3}}\times$ axis of parabola.

[14] If z_1, z_2, z_3 be the depths of the dividing lines, h the axis of parabola; then $z_1=\frac{1}{2}h\times 2^{\frac{1}{3}}$; $z_2=\frac{1}{2}h\times 8^{\frac{1}{3}}$; $z_3=\frac{1}{2}h\times 18^{\frac{1}{3}}$.

[15] $1:2$. [16] Press. on an upper side : press. on a lower $=1:2$.

[17] If ϱ, σ be the densities of water and mercury respectively; Press. on sides : press. on base $=\sigma+3\varrho:\sigma+\varrho$.

[18] Press. on base : press. on each : wt. of water $=3:2\cdot256:1$.

[19] $27:125$. [20] $3:1$. [21] $1:4\cdot523$. [22] $1:3$.

[23] Press. $=\pi\times 122\cdot19$, the density of water being 1.

[24] Depth of circular plane $=$ rad. of sphere $\times 2^{\frac{1}{2}}$.

[25] Depths of dividing planes are $z_1=\frac{1}{2}r$, $z_2=\frac{1}{2}r\times 2^{\frac{1}{2}}$, $z_3=\frac{1}{2}r\times 3^{\frac{1}{2}}$.

[26] Press. : wt. $=r:r-\frac{1}{3}h$; r being $=$ rad. of sphere, $h=$ height of segment.

[27] Depth of the middle point of axis $=\frac{(a^2-b^2)h}{2(a^2+b^2-2nrh)}$.

[28] Press. $=\pi\varrho(2-\cos\alpha)\times$ (rad. of hemisphere)^3.

[29] Press. on base : press. on concave surf. : wt. of fluid $=1:2:1$.

[30] Press. $=\varrho\pi r(r+d)(h+k)$. [31] Depth of plane $=\left(\frac{1}{2}\right)^{\frac{1}{2}}\times$ axis.

[32] $20\cdot7846$ in.; $8\cdot6094$ in.; $6\cdot606$ in.

[33] Ht. $=35$ in.; breadth $=4\cdot9749$ in.

[34] $x_1=\left(\frac{1}{n}\right)^{\frac{1}{2}}h$; $x_2=\left(\frac{2}{n}\right)^{\frac{1}{2}}h$; $x_3=\left(\frac{3}{n}\right)^{\frac{1}{2}}h$; $\ldots x_{n-1}=\left(\frac{n-1}{n}\right)^{\frac{1}{2}}h$; where x_1, $x_2\ldots x_{n-1}$ are the depths of the successive dividing planes.

[35] 10 and 5 in.

[36] Depths of dividing planes are $z_1=\frac{4}{3}(15\times 1)^{\frac{1}{2}}$; $z_2=\frac{4}{3}(15\times 3)^{\frac{1}{2}}$; $z_3=\frac{4}{3}(15\times 7)^{\frac{1}{2}}$.

[37] Density of cylinder $=\frac{3}{2}\times$ density of lighter fluid.

Ex. 1.

[38] Press. $=\frac{1}{3}\pi hr(2l+3r)\rho$. [39] $2:1$.

[40] Depth of dividing plane is a root of $4z^3-6hz^2+h^3=0$.

[41] $z=h\left(\frac{\sigma}{\rho}\right)^{\frac{1}{3}}$. [42] 60°.

[43] $4\{(3h-2m)(h+m)^{\frac{3}{2}}+2m^{\frac{5}{2}}\}:15m^{\frac{1}{2}}h^2$; where h is the height and $4m$ the latus rectum of paraboloid.

[44] $y^2=\frac{c}{h-x}$ is the equation to the generating curve, h being the length of the axis, and c a constant.

Ex. 2.

[1] If θ be the $\angle$ at the centre subtended by the horizontal line,

$$4\sin^3\frac{1}{2}\theta-3\sin\theta+3\theta=3\pi.$$

[2] $\frac{\pi a^3\rho}{8\sqrt{2}}$. [3] Depth of section $=\frac{2}{3}\times$ diam. of sphere; $8:12:15$.

[4] Inclination of axis of hemisphere to the vertical $=\tan^{-1}3$.

[5] $\frac{1}{2}a$.

[6] If h, k be any two depths, the pressures are as $h^{n+1}:k^{n+1}$.

[7] Depth $=\left(\frac{3}{4}\right)^{\frac{1}{2}}\times$ radius.

[8] $\frac{a^3b\sin^2\alpha}{24}\rho$; where $a=$ ht., $b=$ base of Δ, and $\rho=$ density at the depth unity.

[9] Depth of ordinate $=\frac{4}{5}\times$ axis.

[10] $\frac{5}{8}\pi r^3\times$ density at the centre of circle.

[11] $\frac{59}{24}\pi r^3\sigma$; r being the rad. of generating circle, and σ the density at the middle point of the axis of cycloid.

[12] Rad. $=\frac{\sigma}{\rho}$; σ being density of sphere, and ρ the density of the fluid at depth unity.

[13] $\frac{1}{6}ch^2\rho$; $\frac{1}{6}c\frac{d'^3-d^3}{h}\rho$; where c is the circumference of cylinder, and h its length; ρ the density at the lowest points of the fluid.

Ex. 2.

[14] $2\tan^{-1}\frac{1}{2}$. [15] Depth of section $=\frac{3}{5}\times$ axis of cone.

[16] Press. : wt. $=2:(n+1)\tan\alpha$; the vertical $\angle$ of cone being 2α.

[17] Depth of section $=9$; Press. $=\frac{1}{2}\pi k\times 9^3$, where $k=$ density at depth unity.

Centre of Pressure.

Ex. 3.

[1] $\bar{x}=\frac{2}{3}\times$ immersed side. [2] $\bar{x}=\frac{2}{3}\cdot\frac{k^3-h^3}{k^2-h^2}$.

[3] Depth of cent. of gr. $=20$ in. [4] $\bar{x}=\frac{a}{12}\cdot\frac{7a+6\sqrt{2h}}{a+\sqrt{2h}}$.

[5] $\bar{x}=\frac{1}{2}b$; $\bar{y}=\frac{1}{4}a$. [6] $\bar{x}=\frac{h}{2}\cdot\frac{a+3b}{a+2b}$. [7] $\bar{x}=\frac{5}{4}r$.

[8] $\bar{x}=\frac{5}{4}r$; $\bar{y}=\frac{4r}{3\pi}$. [9] $\bar{x}=\frac{3}{16}\pi r$; $\bar{y}=\frac{3}{8}r$.

[10] $\bar{x}=\frac{3}{8}r\left(\frac{\alpha\cos\frac{\alpha}{2}}{\sin\alpha}+\cos\frac{\alpha}{2}\right)$; $\bar{y}=0$; α being the $\angle$ at the centre of the sector.

[11] $\bar{x}=\frac{3k}{7}\left(\frac{5k+7a}{3k+5a}\right)$; where $4a=$ lat. rect. and $k=$ axis.

[12] If the lowest point of cube be taken as the origin of coordinates, the axes being two sides of a lower face, then $\bar{x}=\frac{11}{24}a=\bar{y}$; a being the length of a side.

[13] Depth of hoop $=\frac{2}{3}\times$ length of tub. [14] $\left(\frac{1}{2}\right)^{\frac{1}{2}}r$.

[15] $\bar{x}=\frac{1}{4}r$; $\bar{y}=r\left(\frac{1}{\pi}+\frac{3}{16}\right)$; resultant $=\frac{\sqrt{13}\times\pi r^3}{3}$.

[16] If a be the rad. of cylinder, α the inclination of the axis to the vertical, c^3 the given volume of fluid; then the distance of the cent. of press. from the geometrical centre of base $=\frac{\pi a^4}{4c^3}\tan\alpha$.

[17] $\bar{x}=\frac{4}{5}b$. [18] $\bar{x}=\frac{32r}{15\pi}$. [19] $\bar{x}=\frac{7}{9}h$.

Ex. 4.

[1] If $\theta=$ circular dist. of the common surf. from the lowest point, $\alpha=$ circular length of each fluid; $\tan\theta=\frac{n-1}{n+1}\tan\frac{\alpha}{2}$.

Ex. 4.

[2] $\tan\theta = 2\left(\dfrac{\rho\sin^2\frac{\alpha}{2} - \sigma\sin^2\frac{\beta}{2}}{\rho\sin\alpha + \sigma\sin\beta}\right)$; θ being measured as in the last problem, and $\alpha > \beta$.

[3] If x_1 x_2 measure the distances of the upper surfaces of the heavier and lighter fluids respectively from the lowest point of cycloid; $x_1 = \dfrac{a}{16}\left(\dfrac{3\rho+\rho'}{\rho+\rho'}\right)^2$; $x_2 = \dfrac{a}{16}\left(\dfrac{\rho+3\rho'}{\rho+\rho'}\right)^2$.

Specific Gravity.

Ex. 5.

[1] 252·703 gr. [2] 81 : 500. [3] $66\frac{2}{3}$: 3.

[4] 1·1655 : 1. [5] 3 : 2. [6] 4·51 in.

[7] Diam. $= \left\{\dfrac{6}{\pi}\cdot\dfrac{W-w}{\rho-a}\right\}^{\frac{1}{3}}$; $\sigma = \dfrac{W\rho - wa}{W-w}$. [8] ·643 : 1.

[9] $3\frac{1}{2}$ gr. [10] 11 : 9. [11] 17·55.

[12] 38·803 lb. [13] 387·76 gr. [14] ·800245.

[15] ·8421. [16] 1·6657. [17] ·933.

[18] The mixture is of equal volumes of the two fluids.

[19] 5·619. [21] Sp. gr. $= \dfrac{n}{n-1}\left(\dfrac{v\rho + v'\rho'}{v+v'}\right)$.

[22] 5 : 6. [23] ·9076. [24] ·493 in.

[25] The 1st and 3d globes are placed at distances 2, 1 respectively on one side, and the 2d globe at dist. 3 on the other side.

[26] $\dfrac{1}{27}(19\rho + 8\rho')$. [27] $\dfrac{1}{2}\times$ wt. of hydrometer.

[28] 144 : 73. [29] 422·802 tons.

[30] The depth is a root of $x^3 - 15x^2 + 375 = 0$, which lies between 6·7 and 6·8 in.

[31] 14·512 in. [32] $\dfrac{8}{11}\times$ axis.

[33] Depth $= \dfrac{a+b}{ma+nb}\times$ axis of cylinder.

[34] $\left\{\left(\dfrac{\sigma-\rho}{\sigma-a}\right)^{\frac{1}{3}} - 1\right\}\times$ internal radius.

[35] $r = 9·4$ in. [36] $2·5 \times W$. [37] 4·5 : 4·4944875.

[38] The depth is a root of $x^3 - 1·5dx^2 + ·35895d^3 = 0$; where d = diameter.

Ex. 5.

[39] If ρ, ρ', σ be the Sp. Gr. of the lead, wood, and sea-water respectively, t the given time; then accel. force downwards $(f)=\left(1-\frac{2\sigma}{\rho+\rho'}\right)g$, and upwards $(f')=\left(\frac{\sigma}{\rho'}-1\right)g$:
depth $=\frac{1}{2}t^2\div\left(\frac{1}{f}+\frac{1}{f'}+\frac{2}{\sqrt{ff'}}\right)$.

Equilibrium of Floating Bodies.

Ex. 6.

Let M *be the metacentre,* H, G *the centres of gravity of the fluid displaced, and of the solid respectively*; ρ, σ *the densities of the solid and of the fluid.*

[1] HM $=\frac{1}{12}$ (line of floatation)$^3\div$area of Δ^r part immersed.

[3] The side 6 will be immersed to the extent of 4·8 or 6·2.

[5] HM is bisected by surface of fluid.

[6] $\frac{3}{4}\times$ density of fluid. [7] Neutral.

[8] Altitude of the cone of fluid = (radius of sphere) $\times 2^{\frac{1}{3}}$.

[9] Stable.

[10] HM $=\frac{3a^2}{4h}\left(\frac{\rho}{\sigma}\right)^{\frac{1}{3}}$; a being the rad. of base and h the alt. of cone.

[11] 60°.

[12] Assuming the motion to be about a principal axis of the elliptic section; the equilibrium is stable, neutral or unstable according as the *other* principal axis is greater than, equal to, or less than $\sqrt{2}\times$ (axis of cylinder).

[13] Vertical $\angle = 2\tan^{-1}\left\{\left(\frac{\sigma}{\rho}\right)^{\frac{1}{3}}-1\right\}^{\frac{1}{2}}$. [14] $\frac{1}{8}\times$ density of fluid.

[15] Depth $=\left(\frac{1}{3}\right)^{\frac{1}{2}}\times$ axis.

[16] AM = 10·7735, A being the vertex of paraboloid.

[17] If a, b be the horizontal and vertical sides of rectangle, then MG $=\frac{a^2}{12b}\cdot\frac{\sigma}{\rho}-\frac{b}{2}\left(1-\frac{\rho}{\sigma}\right)$. [18] $\frac{\text{axis}}{\text{lat. rect.}}=\frac{3}{2}$.

Elastic Fluids.

Ex. 7.

[2] 7·206 ft. [3] 11·22 in.

Ex. 7.

[4] If a be the length of the tube, h=alt. of mercury in the barometer; $\frac{1}{2}\left\{a-h\pm\sqrt{(a+h)^2-\frac{4ah}{n}}\right\}$ measures the quantity of mercury required.

[5] 18·99 in. [6] Depth of upper end $=95$ ft.

[7] $x^4-(h+r)x^3-3r^2x^2+(5r^3+3hr^2)x-2r^4=0$,
where r = the rad. and h=ht. of barometric column.

[8] 94 ft.

[9] Density $=\left(1+\frac{h+a-k}{33}\right)\left(1-\frac{k}{a}\right)^2\alpha$;
where a=axis of paraboloid, and α density of atmosphere.

[10] 1·1 in. nearly.

[11] If r, r_1 be the radii, and a, c the altitudes of the open and closed tubes respectively, h the alt. of the water barometer and x the descent; then

$$r^2(r^2+r_1^2)x^2-r_1^2\{(a+c+h)r^2+cr_1^2\}x+acr_1^4=0.$$

[12] 3554 ft. [13] 19227 ft.

[14] If W be the wt. of the material and appendages including the given wt., then $z=60345\log\left(\frac{4\pi r^3}{3W+4\pi r^3\alpha}\right)$.

[15] Additional ht. $=60345\log\left\{\frac{3W+4\pi r^3\alpha}{3(W-w)+4\pi r^3\alpha}\right\}$ ft.;

barometer sinks $\frac{3wh}{3W+4\pi r^3\alpha}$ in.

[16] 16·44 ft. [17] 14623 ft.

INSTRUMENTS AND MACHINES.

Ex. 8.

[1] $\frac{10}{9}$ in. [2] 40·473 ft. [3] 24·5278 in.

[4] $c+\frac{l-a}{l-c}(h-a)$.

[5] If x, y be the lengths of tube which the air left in them, of the mean density of the atmosphere, would occupy,

$$x=\left\{\frac{a'(l+a-a')}{a}-\frac{b'(l+b-b')}{b}\right\}+\left\{\frac{a'(l-a')}{a(l-a)}-\frac{b'(l-b')}{b(l-b)}\right\};$$

$$y=\left\{\frac{a(l+a'-a)}{a'}-\frac{b(l+b'-b)}{b'}\right\}+\left\{\frac{a(l-a)}{a'(l-a')}-\frac{b(l-b)}{b'(l-b')}\right\}.$$

[6] $a+\frac{h-h''}{h-h'}(a'-a)$. [7] $1:4^{\frac{1}{3}}-1$ or $1:\cdot5874$.

Ex. 8.

[8] 1 : ·4443. [9] 9 : 10 nearly.

[10] Wt. $=\dfrac{a'(\mathrm{R}+b)^n-a\mathrm{R}^n}{(\mathrm{R}+b)^n-\mathrm{R}^n}$;

Density req$^{\text{d}}$ $=\dfrac{(a'-a)(\mathrm{R}+b)^n}{a'(\mathrm{R}+b)^n-a\mathrm{R}^n}\times$ Density of body.

[11] No. of turns $=\log\left(1-\dfrac{q}{p}\right)\div\log\dfrac{\mathrm{R}}{\mathrm{R}+b}$. [12] 25 : 1·21.

[13] No. of turns $=\log\left(1-\dfrac{k}{h}\right)\div\log\dfrac{\mathrm{R}}{\mathrm{R}+b}$;
where h = ht. of barometer.

[14] If h = standard alt., and x = length of tube occupied by the air, when of the atmospheric density; then

$$h=a\left(\frac{c}{a}-\frac{l-a}{l-c}\right)\div\left\{\left(\frac{m}{m+1}\right)^n-\frac{l-a}{l-c}\right\};$$

and $x=(l-a)\left\{\dfrac{c}{a}-\left(\dfrac{m}{m+1}\right)^n\right\}\div\left(\dfrac{c}{a}-\dfrac{l-a}{l-c}\right)$.

[15] 1 : ·68558, or 1·4586 : 1. [16] $\left\{2-\left(\dfrac{\mathrm{R}}{\mathrm{R}+b}\right)^n\right\}\varrho$.

[17] $\left\{\dfrac{a}{c}+\left(1-\dfrac{a}{c}\right)\left(\dfrac{\mathrm{R}}{\mathrm{R}+c}\right)^n\right\}\varrho$. [18] $5\frac{5}{7}$ in. from the receiver.

[19] The successive alts. are 1·16, 2·2318, 3·2274, 4·1485, 5 in.

[20] 4·694 in.

[21] General term of the series is $x_r=a\left\{1-\left(\dfrac{\mathrm{R}}{\mathrm{R}+b}\right)^r\right\}$.

[22] $(h'-h)\left(\sigma+\dfrac{a}{h'}\right)$. [23] 72·6 ft.

[24] $\left(\dfrac{h+k}{nh}-1\right)\mathrm{V}$; h being = alt. of water barometer.

[25] (1) The water will pass through the valve in the piston; (2) 9 or 11 ft.

[27] 20 : 36 : 45. [28] $m=20$; $n=16$. [29] $88°\frac{1}{4}$.

[30] 3·054 cub. in. [31] 68°. [32] ·0043859. [33] $\dfrac{c-d}{a-b}n$.

HYDRODYNAMICS.

Ex. 9.

[1] $17\frac{7}{9}$ ft. [2] Vel. = 25·7 ft.; ht. = 10·3 ft.

[3] $5\frac{1}{12}$ ft.; $7\frac{1}{16}$ ft.; $11\frac{1}{3}$ ft. [4] $6\frac{5}{11}$ ft.

[5] On the horizontal plane of the base, at a distance from the side of the vessel = 11·832 ft.

Ex. 9.

[6] Range, measured from the lowest point of cyl. $= 13{\cdot}478$ ft.

[7] $1\frac{3}{8}$ ft. [8] Depth of orifice is a root of $2x^{\frac{3}{2}} - 4x^{\frac{1}{2}} = 1$.

[9] $1{\cdot}4 \times 2^{\frac{1}{2}}$ ft.

[10] If $\theta =$ inclination of the radius through the orifice with the vertical; then $(1 + \sin\theta)^2 = 4\sin\theta\,(\cos\theta + \sin^2\theta)$.

[11] Depth $= \frac{1}{2} \times$ length of cyl. [12] Depth of orifice $= 3{\cdot}17$ ft.

[13] Vertical depth of orifice $= 1{\cdot}43$ ft.

[14] $1 : 2$. [15] $4 : 1$.

[16] Depth of dividing plane $= 10\frac{1}{2}$ ft. [17] $16 : 1$.

[18] $\frac{K}{k}\left(\frac{2l}{g\sigma}\right)^{\frac{1}{2}} \cdot \left\{(\sigma + 1)^{\frac{1}{2}} + \sigma^{\frac{1}{2}} - 1\right\}$; K being $=$ area of base of cyl.

[19] $\pi^2 : 8$. [20] 31 hr. 25 m. 37 sec.

[21] If x, d be the diameters of the orifice and cylinder, $a =$ altitude of cylinder; $x^2 = \frac{d^2}{t}\left\{1 - \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\right\}\left(\frac{2a}{g}\right)^{\frac{1}{2}}$.

[22] Rad. of cyl. $= a\left\{2bg\left(\frac{s-r}{m-n}\right)^2\right\}^{\frac{1}{4}}$; ht. $= \frac{1}{4b}\left(\frac{ms-nr}{s-r}\right)^2 + b$.

[23] If $h =$ ht. of cone, $r =$ rad. of base; $t = \frac{\pi r^2}{k}\left(\frac{h}{g}\right)^{\frac{1}{2}} \cdot \frac{2^{\frac{5}{2}} - 1}{20}$.

[24] $3 : 8$. [25] $\frac{1246}{15k} \cdot \frac{\pi}{\sqrt{g}}$. [26] $\frac{a^2}{5k}\left(\frac{2h}{g}\right)^{\frac{1}{2}}$.

[28] $\frac{7\pi r^2}{15k}\left(\frac{2r}{g}\right)^{\frac{1}{2}}$. [29] $\frac{4\pi r^2}{5k}\left(\frac{2r}{g}\right)^{\frac{1}{2}}$. [30] $\left(\frac{36}{35}\right)^{\frac{2}{5}}, = 1{\cdot}0113$.

[31] $275 : 343$.

[32] $\frac{\pi hl}{3k}\left(\frac{2h}{g}\right)^{\frac{1}{2}}$; if $h =$ axis, and $l =$ lat. rect. of paraboloid.

[33] $1 : 2$. [34] $y = Ax^2$. [35] $b^{\frac{1}{2}} : a^{\frac{1}{2}}$.

[36] $\frac{16\pi ab}{15k}\cdot\left(\frac{c}{g}\right)^{\frac{1}{2}}$. [37] $\frac{\pi a^2}{k}\left(\frac{a}{g}\right)^{\frac{1}{2}}\left(2\pi^2 - \frac{8^3}{45}\right)$. [38] $y^4 = Ax$.

[39] $55{\cdot}008$ ft. [40] $1335{\cdot}17$ ft.

[41] If h ft. of water measure the atmospheric pressure, $a =$ axis, Depth of orifice $= h + \frac{a}{\sqrt{2}}$, vel. $= \left\{2g\left(h + \frac{a}{\sqrt{2}}\right)\right\}^{\frac{1}{2}}$.

[42] $1250{\cdot}45$ ft. [43] $1831{\cdot}355$ ft.

RESISTANCES.

Ex. 10.

[1] $2:3$.

[2] $d^2-\frac{1}{3}c^2:d^2$; d, c being the diam and chord respectively.

[3] $\frac{a^2b}{(a^2-b^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{a^2}{b^2}-1\right)^{\frac{1}{2}}-\frac{b^2}{a^2-b^2}$.

[4] $\tan^{-1}\frac{b}{2m}:\frac{b}{2m}$; where b is the extreme ordinate, and $4m$ the lat. rect.

[5] $3:4$.

[6] $1:2$.

[7] $r^2-\frac{1}{2}c^2:r^2$; r, c being the radii of sphere and segment's base respectively.

[8] $\frac{3\pi d}{8l}$; d, l are the diam. of base and length of cylinder respectively.

[9] $\sin^2\alpha:1$; 2α being the vertical $\angle$ of cone.

[10] $\frac{L^2}{b^2}\log\left(1+\frac{b^2}{L^2}\right)$; where $L=$ lat. rect., $b=$ diam. of base.

[11] If $2a$, $2b$ be the axes of the generating ellipse, v the vel. of the sphere, and ϱ the density of the fluid; then
the resistance $=\frac{\pi\varrho v^2b^4}{2(a^2-b^2)}\left(\frac{2a^2}{a^2-b^2}\log\frac{a}{b}-1\right)$.

[12] $9\pi^2-16:12\pi^2$.

THE END.

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